Summary Notes

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Theory of Condensed Matter

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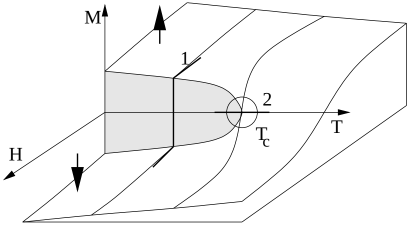
Topology of magnetic transition phase diagram

Typical Phase Diagram: Magnetism



- Phase transitions may be discontinuous (path 1) or continuous (path 2).
- Phases distinguished by order parameter M.

Definitions of Critical exponents

Critical behaviour near T_c:

$$M \sim (-t)^{\beta} \text{ at } H = 0, t < 0$$
$$\chi \ \left(= \frac{\partial M}{\partial H}\Big|_{H=0}\right) \sim |t|^{-\gamma} \text{ at } H = 0$$
$$H \sim |M|^{\delta} \operatorname{sgn} M \text{ at } t = 0$$
$$C_H \sim |t|^{-\alpha} \text{ at } H = 0$$
$$\xi \sim |t|^{-\nu}$$
$$G(r) \sim \frac{1}{r^{d-2+\eta}} \text{ at } t = 0$$

Near *critical point*, microscopic length scales should not play a fundamental role $\rightsquigarrow$ phenomenological description.

Critical Phenomena and Ginzburg-Landau Theory

- Divergence of correlation length ξ motivates construction of phenomenological theory based on fundamental symmetries.
- Ginzburg-Landau Hamiltonian

$$\beta H = \int d\mathbf{x} \left[\frac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + \cdots + \frac{K}{2}(\nabla\mathbf{m})^2 + \cdots - \mathbf{h}\cdot\mathbf{m} \right].$$

- Assumed to arise from integrating over short-length fluctuations.
- Partition Function

$$\mathcal{Z} = \int D\mathbf{m}(\mathbf{x}) e^{-\beta H[\mathbf{m}]}.$$

Landau MFT

$$\mathcal{Z} = e^{-\beta \overbrace{F[h,T]}^{\text{Free energy}}} \underbrace{\sim}_{\text{s.p.a.}} e^{-\min_{\mathbf{m}} \beta H[\mathbf{m}]}$$

For $K > 0$, min. when $\mathbf{m}(\mathbf{x}) = \bar{m}\mathbf{e_h}$ i.e.

$$\frac{\beta F}{V} = \min_{\mathbf{m}}[\tfrac{t}{2}m^2 + um^4 - hm].$$

Use to infer

$$t = \frac{T - T_c}{T_C}$$

and critical exponents: $\beta = 1/2$, $\delta = 3$, $\gamma = 1$, and $\alpha = 0$.

Gaussian Functional Integrals

$$\mathcal{Z} = \int D\phi(\mathbf{x}) \ \exp\left[-\frac{1}{2}\int d^d\mathbf{x}\left(\frac{\phi^2}{\xi^2} + (\nabla\phi)^2\right)\right]$$

- $\langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle_c = G(\mathbf{x},\mathbf{x}')$ where

$$\left(-\nabla'^2 + \xi^{-2}\right) G(\mathbf{x},\mathbf{x}') = \delta^d(\mathbf{x}-\mathbf{x}').$$

- Equivalently

$$G(\mathbf{q}) = \frac{1}{\mathbf{q}^2 + \xi^{-2}}$$

  and

$$\langle\phi(\mathbf{q})\phi(\mathbf{q}')\rangle_c = (2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')G(q).$$

- If $\mathcal{A} = \int d^d\mathbf{x} a(\mathbf{x})\phi(\mathbf{x})$ then

$$\langle e^{\mathcal{A}}\rangle = e^{\langle\mathcal{A}\rangle_c + \langle\mathcal{A}^2\rangle_c/2}$$

  Proof: *Complete the square.*

Derivation of Ginzburg-Landau Hamiltonians from microscopic models

- Introduce an order parameter Ψ via Hubbard-Stratonovich decoupling (a.k.a. reversing completing the square)

$$\mathcal{Z} = \det\left[2\pi G_{ij}^{-1}\right]^{-\frac{1}{2}} \sum_{\{\sigma_i=\pm 1\}} \int \prod_i d\Psi_i e^{-\frac{1}{2}\sum_{ij} G_{ij}^{-1}\Psi_i\Psi_j} e^{\sum_i(\Psi_i+h)\sigma_i}$$

- Integrate out the original microscopic degrees of freedom
- Re-exponentiate to obtain a GL Hamiltonian
- Expand in powers of the order parameter and its gradients close to the critical point
- Read off from the coefficients the relevant t, u, $\ldots$, phenomenological parameters.

Continuous Symmetry Breaking and Goldstone Modes

- Ginzburg-Landau Hamiltonian

$$\beta H = \int d\mathbf{x} \left[\frac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + \frac{K}{2}(\nabla\mathbf{m})^2\right].$$

- Landau Mean-Field:
  $t < 0$, Spontaneous symmetry breaking $\rightsquigarrow$ appearance of ordered ground state
- Breaking of a continuous symmetry $\rightsquigarrow$ low-energy excitations (Goldstone Modes)
  - magnet — spin waves
  - crystal lattice — phonons

Goldstone modes effect on Long Range Order

- Transverse Fluctuations $\mathbf{m}(\mathbf{x}) = \bar{m}(\cos\theta(\mathbf{x}), \sin\theta(\mathbf{x}))$
- Neglecting topological (vortex) configurations

$$\langle \mathbf{m}(\mathbf{x}) \cdot \mathbf{m}(0) \rangle = \bar{m}^2 \exp\left[-\frac{1}{2} \langle [\theta(\mathbf{x}) - \theta(0)]^2 \rangle \right]$$

$$\xrightarrow[|\mathbf{x}|\to\infty]{} \begin{cases} \bar{m}'^2, & d > 2 \\ 0, & d \leq 2. \end{cases}$$

- Mermin-Wagner Theorem:
  For systems with a continuous symmetry (and short ranged interactions) there is no LRO in dimensions $d \leq 2$ — the lower critical dimension.

Role of Fluctuations in GL Theory

$$\beta H = \int d\mathbf{x} \left[\frac{t}{2}\mathbf{m}^2 + u\mathbf{m}^4 + \frac{K}{2}(\nabla\mathbf{m})^2\right]$$

Parametrise

$$\mathbf{m}(\mathbf{x}) = [\bar{m} + \phi_l(\mathbf{x})]\,\hat{\mathbf{e}}_1 + \sum_{\alpha=2}^{n} \phi_{t,\alpha}(\mathbf{x})\hat{\mathbf{e}}_\alpha$$

and expanding to second order

$$\beta H[\mathbf{m}(\mathbf{x})] = \underbrace{\beta H[\bar{m}]}_{\text{Landau MFT}} + \int d\mathbf{x}\, \frac{K}{2} \sum_{\alpha=l,t} \left[(\nabla\phi_\alpha)^2 + \xi_\alpha^{-2}\phi_\alpha^2\right]$$

Correlation Function:

$$\langle \phi_\alpha(\mathbf{q})\phi_\beta(\mathbf{q}')\rangle_c = \delta_{\alpha\beta}\ (2\pi)^d\ \delta^d(\mathbf{q}+\mathbf{q}') \times \frac{1}{K(\mathbf{q}^2 + \xi_\alpha^{-2})}$$

Effect of Fluctuations

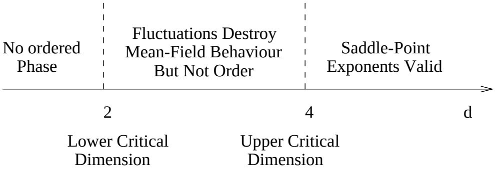
Real space:

$$\langle \phi_\alpha(\mathbf{x})\phi_\beta(\mathbf{x}')\rangle_c = G_{\alpha\beta}(\mathbf{x},\mathbf{0})$$
$$\sim \begin{cases} C_d(\mathbf{x}) = \frac{|\mathbf{x}|^{2-d}}{(2-d)S_d} & |\mathbf{x}| \ll \xi, \\ \frac{\xi^{2-d}}{(2-d)S_d}\ \frac{\exp[-|\mathbf{x}|/\xi]}{|\mathbf{x}/\xi|^{(d-1)/2}} & |\mathbf{x}| \gg \xi. \end{cases}$$

i.e. ξ is the correlation length

$$\xi \sim \left(\frac{K}{t}\right)^{\frac{1}{2}}.$$

Summary of Landau Theory



- $d < 4$: Beyond saddle-point analysis gives divergent corrections to thermodynamics quantities, response functions and correlation length.
- But can only see deviations from mean field results if experiment can resolve beyond Ginzburg criterion

$$t_G \approx \frac{1}{[(\xi_0/a)^d(\Delta C_{\text{sp}}/k_B)]^{2/(4-d)}}.$$

Scaling Hypothesis

Assuming correlation length takes a homogeneous form

$$\xi(t,h) \sim t^{-\nu} g_\xi\left(\frac{h}{t^\Delta}\right)$$

and, close to T_c, is the only important length scale implies

- Free energy (and other thermodynamic quantities) also takes homogeneous form

$$f_{\text{sing.}}(t,h) = t^{2-\alpha} g_f\left(\frac{h}{t^\Delta}\right).$$

- Two independent exponents fix all other critical exponents. Examples include

$$\alpha + 2\beta + \gamma = 2, \qquad \text{(Rushbrooke's Identity)}$$

$$\delta - 1 = \gamma/\beta. \qquad \text{(Widom's Identity)}$$

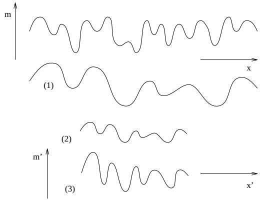
Consequences of Scaling

- Critical system has an additional **dilation symmetry**.
- Under a change of scale, the critical correlation functions behave as

$$G_{\text{critical}}(\lambda \mathbf{x}) = \lambda^{p} G_{\text{critical}}(\mathbf{x}).$$

- Statistical self-similarity cannot be directly implemented in Ginzburg-Landau scheme of symmetries and contraints.
- Progress using less direct route: the **renormalisation group**.

Kadanoff's Renormalisation Group (conceptual)



Start with a configuration $\mathbf{m}(\mathbf{x})$ with weight $W[\mathbf{m}] = e^{-\beta H[\mathbf{m}]}$.

1. Coarse-grain:

$$\bar{\mathbf{m}}(\mathbf{x}) = \frac{1}{(ba)^d} \int_{\text{Cell}} d\mathbf{y}\ \mathbf{m}(\mathbf{y}).$$

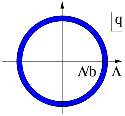
2. Rescale:

$$\mathbf{x}' = \frac{\mathbf{x}}{b}.$$

3. Renormalise:

$$\mathbf{m}'(\mathbf{x}') = \frac{1}{\zeta}\bar{\mathbf{m}}(\mathbf{x}').$$

RG applied to Gaussian Model



Coarse-Grain Eliminate fluctuations at scales $a < |\mathbf{x}| < ba$ or removal of Fourier modes $\Lambda/b < |\mathbf{q}| < \Lambda$. Separate the fields into slowly and rapidly varying functions, $\mathbf{m}(\mathbf{q}) = \mathbf{m}_>(\mathbf{q}) + \mathbf{m}_<(\mathbf{q})$

Integrate over fast variables Partition function becomes

$$\mathcal{Z} = \mathcal{Z}_> \int D\mathbf{m}_< \exp\left[-\int_0^{\Lambda/b} (d\mathbf{q}) \left(\frac{t + K\mathbf{q}^2}{2}\right) |\mathbf{m}_<|^2 + \mathbf{h} \cdot \mathbf{m}_<(0)\right].$$

Gaussian Model RG 2

Rescale $\mathbf{x}' = \mathbf{x}/b$ in real space, or $\mathbf{q}' = b\mathbf{q}$ in momentum space to restore cut-off.

Renormalise $\mathbf{m}'(\mathbf{x}') = \mathbf{m}_<(\mathbf{x}')/\zeta$ or $\mathbf{m}'(\mathbf{q}') = \mathbf{m}_<(\mathbf{q}')/z$ giving

$$\mathcal{Z} = \mathcal{Z}_> \int D\mathbf{m}'(\mathbf{q}')e^{-\beta H'[\mathbf{m}'(\mathbf{q}')]},$$

$$\beta H' = \int_0^\Lambda (d\mathbf{q}) b^{-d} z^2 \left(\frac{t + Kb^{-2}\mathbf{q}'^2}{2} \right) |\mathbf{m}'|^2 - z\mathbf{h} \cdot \mathbf{m}'(0).$$

Results

$$\begin{cases} t' = b^2 t & y_t = 2, \\ h' = b^{1+d/2} h & y_h = 1 + d/2. \end{cases}$$

Both relevant ($y_t > 0$ and $y_h > 0$).

Gaussian Model RG 3

Adding a term $u \int d^d\mathbf{x}\ m^4$ gives

$$u' = b^{4-d}u.$$

In $d > 4$ u provides an irrelevant perturbation but in $d < 4$, it is relevant (grows under RG). We must therefore include u in RG.

Wilson's Perturbative RG

$$\beta H = \underbrace{\beta H_0}_{\text{Gaussian part}} + \underbrace{U}_{\text{Perturbation}}$$

$$\beta H_0 = \int (d\mathbf{q}) \frac{G_0^{-1}}{2} |\mathbf{m}(\mathbf{q})|^2$$

Fourier representation of perturbation

$$\begin{aligned} U &= u \int d^d\mathbf{x}\ (\mathbf{m} \cdot \mathbf{m})^2 \\ &= u \int (d\mathbf{q}_1)(d\mathbf{q}_2)(d\mathbf{q}_3)(d\mathbf{q}_4)\mathbf{m}(\mathbf{q}_1) \cdot \mathbf{m}(\mathbf{q}_2)\ \mathbf{m}(\mathbf{q}_3) \cdot \mathbf{m}(\mathbf{q}_4) \\ &\times (2\pi)^d \delta^d(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4) \end{aligned}$$

Wilson's Perturbative RG II

In the partition function

$$\mathcal{Z} = \int D\mathbf{m}_<(\mathbf{q}) D\mathbf{m}_>(\mathbf{q})$$
$$\exp\left\{ - \int_0^\Lambda \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) \left(|m_<(\mathbf{q})|^2 + |m_>(\mathbf{q})|^2 \right) - U \right\}$$
$$= \int D\mathbf{m}_<(\mathbf{q}) \mathbf{e}^{-\beta H'}$$

the two sets of modes are mixed by the operator U and

$$\beta H'[m_<] = V\delta f_b^0 + \int_0^{\Lambda/b} \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) |m_<(\mathbf{q})|^2 - \log \langle \mathrm{e}^{U[m_<, m_>]} \rangle_{m_>}$$

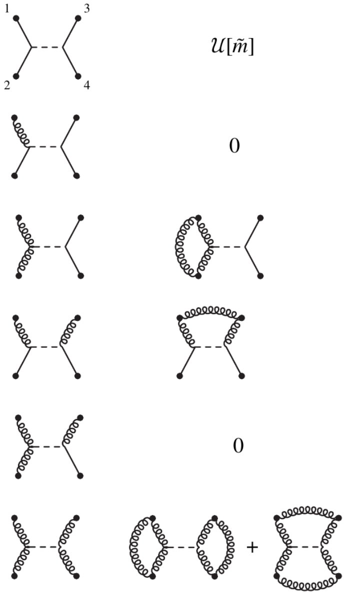
Wilson's Perturbative RG III

Here the partial averages are defined by

$$\langle \mathcal{O} \rangle_{m_>} \equiv \int \frac{Dm_>(\mathbf{q})}{\mathcal{Z}_>} \mathcal{O} \exp\left\{ -\int_{\Lambda/b}^{\Lambda} \frac{d^d\mathbf{q}}{(2\pi)^d} \left(\frac{t + Kq^2}{2} \right) |m_>(\mathbf{q})|^2 \right\}$$

and $\log\langle \mathrm{e}^{U[m_<, m_>]} \rangle_{m_>}$ is a cumulant generating function and this perturbative expansion can be calculated with the aid of Feynman diagrams.

Wilston's Perturbative RG IV



Perturbative Results

After coarse-graining, coefficients K and u are unchanged, while

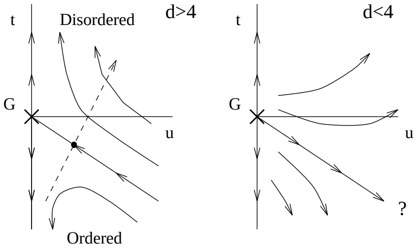
$$t \mapsto \widetilde{t} = t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^d} G_0(\mathbf{q}),$$

the factor of $4(n+2)$ arising from enumerating all permutations. By setting $b = \mathrm{e}^l$, for an infinitesimal δl, we find the recursion relations linearised about the fixed point $t^* = u^* = 0$, by setting $t = t^* + \delta t$ and $u = u^* + \delta u$, as

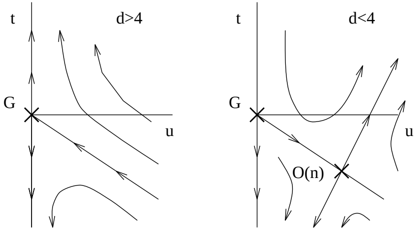
$$\frac{d}{d\ell}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix} = \begin{pmatrix}2 & 4(n+2)K_d\Lambda^{d-2}/K \\ 0 & 4-d\end{pmatrix}\begin{pmatrix}\delta t \\ \delta u\end{pmatrix}.$$

Perturbative Results II

One loop:



Two loop:



Quantum–classical mapping

Quantum:

$$\hat{H} = \sum_{i=1}^{N} \frac{\hat{\mathbf{p}}_i^2}{2m} + \hat{V}(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, ..., \hat{\mathbf{x}}_N),$$

$$\mathcal{Z} = \int \left(\prod_{i=1}^{N} d^d \mathbf{x}_i \right) \langle \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N | e^{-\beta \hat{H}} | \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N \rangle.$$

Classical:

$$\mathcal{Z} = \int_{\mathbf{x}_i(\beta) = \mathbf{x}_i(0)} \mathcal{D}\mathbf{x}_i(\tau) e^{-H[\mathbf{x}_i(\tau)]},$$

$$H[\mathbf{x}_i(\tau)] = \int_0^{\beta} d\tau \left[\sum_{i=1}^{N} \frac{m |\partial_\tau \mathbf{x}_i|^2}{2} + V[\mathbf{x}_i(\tau)] \right].$$

A d-dimensional quantum system at finite temperature β^{-1} can be mapped onto a $(d+1)$-dimensional classical system

Path Integral Representation

$$
\begin{aligned}
\mathcal{Z} &= \int dX \langle X| e^{-\beta \hat{H}} |X\rangle \\
&= \int dX \langle X| e^{-\frac{\beta}{N_\tau}\hat{H}} \mathbf{1} e^{-\frac{\beta}{N_\tau}\hat{H}} \mathbf{1} e^{-\frac{\beta}{N_\tau}\hat{H}} \dots e^{-\frac{\beta}{N_\tau}\hat{H}} |X\rangle
\end{aligned}
$$

Insert resolution of identities with expanded exponentials:

$$
\mathcal{Z} = \int \left(\prod_{i=1}^{N_\tau} dX_i\right) \int \left(\prod_{i=1}^{N_\tau} dP_i\right) \langle X_1|P_1\rangle\langle P_1| \left(1 - \epsilon\hat{H}\right) |X_2\rangle \times
$$

$$
\langle X_2|P_2\rangle\langle P_2| \left(1 - \epsilon\hat{H}\right) |X_3\rangle \times \dots \times \langle X_{N_\tau}|P_{N_\tau}\rangle\langle P_{N_\tau}| \left(1 - \epsilon\hat{H}\right) |X_1\rangle.
$$

$$
\mathcal{Z} = \int_{X(\beta)=X(0)} \mathcal{D}X(\tau) \int \mathcal{D}P(\tau) e^{-\int_0^\beta d\tau \left(iP(\tau)\cdot\partial_\tau X(\tau) + \frac{P^2}{2m} + V[X(\tau)]\right)}
$$

$O(2)$ Rotor

$$\hat{H}_{\mathrm{O}(2)} = \sum_i \frac{\hat{L}_i^2}{2m} - g \sum_{\langle ij \rangle} \hat{\mathbf{x}}_i \cdot \hat{\mathbf{x}}_j,$$

$$\mathcal{Z}_{\mathrm{O}(2)} = \int_{\phi_i(\beta)-\phi_i(0)=2\pi n} \mathcal{D}\phi_i(\tau) e^{-H[\phi_i(\tau)]},$$

$$H[\phi_i(\tau)] = \int_0^\beta d\tau \left[\sum_{i=1}^{N} \frac{m(\partial_\tau \phi_i)^2}{2} - g \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j) \right],$$

At zero temperature ($\beta \to \infty$) the d-dimensional quantum system maps onto $(d+1)$-dimensional classical system with one Goldstone mode with $\omega = \sqrt{g/m}|q|$. Mermin-Wagner theorem: no LRO if $d \leqslant 1$ — verify by expanding about ordered state and calculating $\langle \phi_i^2(0) \rangle$.