Let us consider continuous spins near 2 dimensions
e.g. lattice ferromagnets

$$\beta H = -K \sum_{\langle ij \rangle}' \underline{S}_i \cdot \underline{S}_j \qquad S_i^2 = 1$$

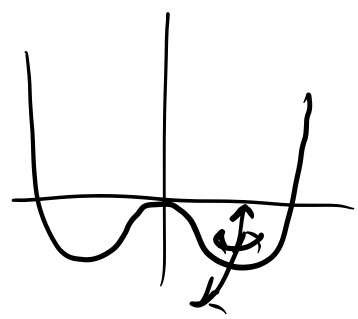
$$\frac{J}{k_B T} > 0$$

Low T $\rightsquigarrow$ continuum limit $\rightsquigarrow$ $\beta H = \frac{K}{2} \int d^d \underline{x} \, (\underline{\nabla} \underline{S})^2$

with $S^2(x) = 1$

Non-linear σ model

$$Z = \int_{S^2(\underline{x}) = 1} \mathcal{D} \underline{S}(\underline{x}) \, e^{-\beta H[S]}$$



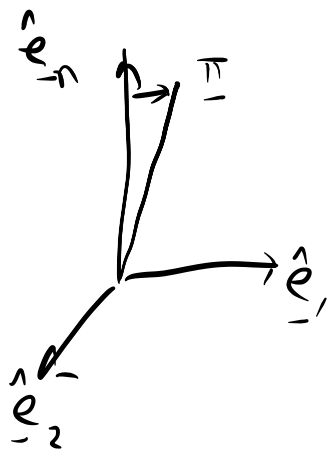
(cf. G-L theory for $t \ll 0$.)

Low T expansion about ground state $(0, 0 \ldots 0, 1)$

$$\underline{S} = (\underbrace{\pi_1, \pi_2, \ldots}_{\pi}, (1 - \pi^2)^{\frac{1}{2}})$$

π – $n-1$ component vector describing deviation from $\hat{e}_n$

$$\beta H = \frac{K}{2}\int d^d \underline{x} \left((\underline{\nabla}\, \underline{\pi})^2 + \left(\underline{\nabla}\left(1-\pi^2\right)^{\frac{1}{2}}\right)^2 \right)$$



Gaussian

candidate for perturbative RG

$$\mathcal{Z} = \int \mathcal{D}\pi \; e^{-\beta H[\pi]}$$

fluctuations

Need expansion → all higher order terms in π are non-linear

Extension exercise!

$2+\varepsilon$ expansion Polyakov 1974

At Gaussian order $\exists$ no LRO for $d \leq 2$ (Mermin-Wagner)

$n=2$ version $\underline{S}$ done previously ✓

1st order RG $\quad K = \frac{1}{T}$

Flow eqn $\quad \frac{dT}{dl} = -(d-2)\,T + (n-2)\, \frac{S_d \Lambda^{d-2}}{(2\pi)^d}\, T^2 + \dots$

If $d=2$, is there a phase transition?

$$\begin{cases} n<2 & \text{Yes - Ising model} \\ n>2 & \text{No} \\ n=2 & \text{?? both terms vanish!} \end{cases}$$

Even if we calculate to higher orders we still find 0! $\Rightarrow$ Something interesting happens.

$n = 2$: XY model

$$\beta H = \frac{K}{2}\int d^2\underline{x}\ |\underline{\nabla}\theta|^2$$

· Low T (done previously) $= K^{-1}$

$$\langle \underline{S}(\underline{x})\cdot\underline{S}(\underline{0})\rangle = \mathrm{Re}\langle e^{i[\theta(x)-\theta(0)]}\rangle$$

$$= e^{-\frac{1}{2}\langle(\theta(x)-\theta(0))^2\rangle}$$

$$= \frac{1}{\pi K}\log\left(\frac{|x|}{a}\right) = \left(\frac{a}{|\underline{x}|}\right)^{\frac{1}{2\pi K}}$$

i.e. power law decay

<u>Quasi</u>-LRO $\Rightarrow \xi = \infty$ length scale at which decay exponential.

no L.R.O. but $\xi = \infty$

High T $K \ll 1$, go back to lattice model

$$Z = \int_0^{2\pi} \prod d\theta_i \, e^{K \sum'_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$$

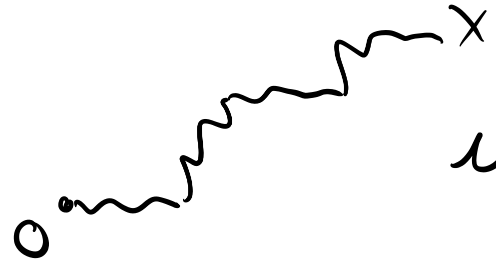
$K \to 0$
$$= \int_0^{2\pi} \prod d\theta_i \prod_{\langle ij \rangle} \left(1 + K \cos(\theta_i - \theta_j) + O(K^2)\right)$$

$$\langle S(x) \cdot S(0) \rangle = \frac{1}{Z} \int_0^{2\pi} \prod_i d\theta_i \cos(\theta_x - \theta_0) \prod_{\langle ij \rangle} \left(1 + K \cos(\theta_i - \theta_j)\right)$$

Trig results $\displaystyle\int_0^{2\pi} d\theta_2 \cos(\theta_1 - \theta_2)\cos(\theta_2 - \theta_3) = \frac{1}{2}\cos(\theta_1 - \theta_3)$

and $\displaystyle\int_0^{2\pi} d\theta_2 \cos(\theta_1 - \theta_2) = 0$

So to obtain a contribution need shortest string



count # of bonds

$\frac{1}{2}$ each time

High T

$$\langle S(x) \cdot S(0) \rangle \simeq \left(\frac{K}{2}\right)^{|x|} = e^{-|x|/\xi}, \quad \xi = \frac{1}{\log(2/K)}$$

exp decay

QLRO Disordered

0 —— T_c ? —— $T \to \infty$

power law correlations

Exponential decay.

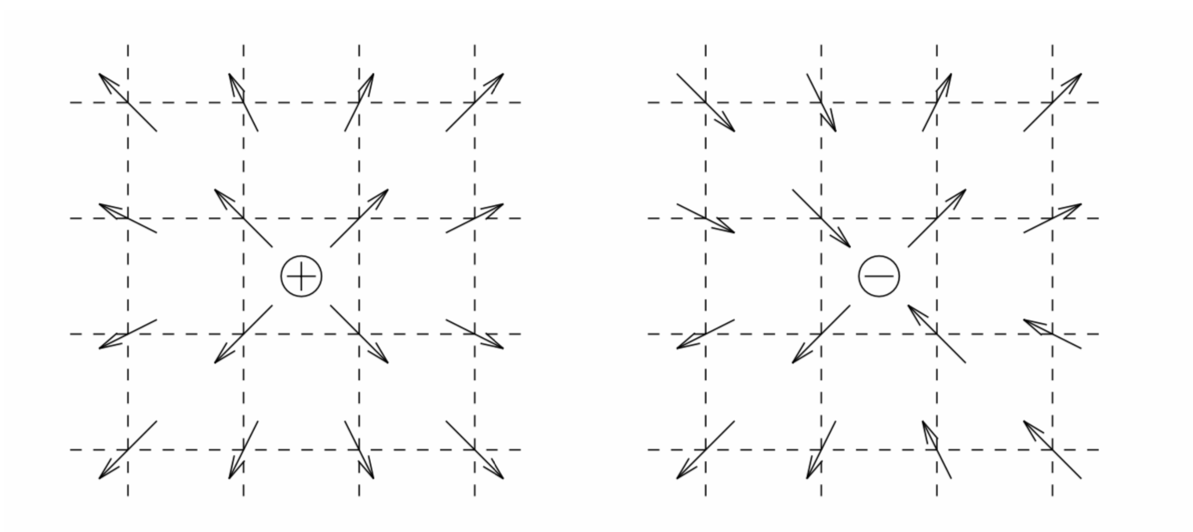
Phase transitions <u>without</u> symmetry breaking?

Berezinskii - Kosterlitz - Thouless transition

1969 1972

Periodicity of θ implies topological defects

$\rightsquigarrow$ Vortices responsible for transition.



2π -2π

Energy costs?

Far away $\cos(\theta_i - \theta_j) \simeq 1 - \frac{1}{2}(\underline{\nabla}\theta)^2 + \dots$

Change in θ circulating vortex

$$\Delta\theta = \oint_C d\underline{x} \cdot \underline{\nabla}\theta = 2\pi n \qquad n \in \mathbb{Z}$$

(cf. Burgers vector in a crystal)

By symmetry $\quad 2\pi r \frac{d\theta}{dr} = 2\pi m \qquad \frac{d\theta}{dr} = \frac{m}{r}$

$$\beta E_m = \frac{K}{2} \int d^2\underline{x} \, |\underline{\nabla}\theta|^2 = \frac{K}{2} \int_a^L 2\pi r \, dr \left(\frac{m}{r}\right)^2$$

$$= \pi K m^2 \log\left(\frac{L}{a}\right) \qquad \text{cf. dislocation}$$

log divergent energy but
what about entropy?

Entropy

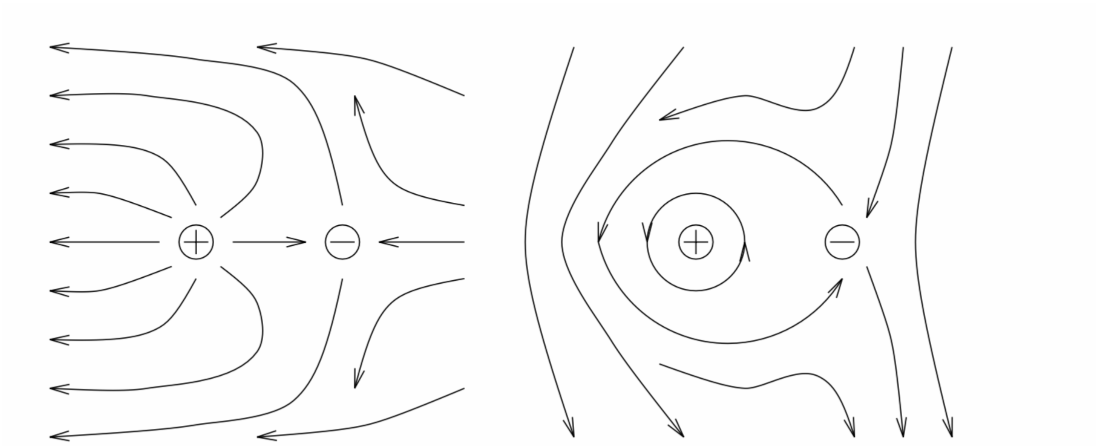
Can set vortex in $\left(\frac{L}{a}\right)^2$ different locations

$$\therefore \quad Z = \left(\frac{L}{a}\right)^2 e^{-\beta E_m} = \left(\frac{L}{a}\right)^{2-\pi K m^2}$$

i.e. condensate of vortices when $K < K_c = \frac{2}{\pi}$ $\quad m = \pm 1$

$\rightarrow$ origin of phase transit

Note pair of vortices have finite energy



note far away spin orientation again parallel.

pair of vortices

$$|\underline{\nabla}\theta| = |\underline{\nabla}\theta_+ + \underline{\nabla}\theta_-| = \underline{l} \cdot \underline{\nabla}\left(\frac{1}{|\underline{x}|}\right) \sim \frac{l}{x^2}$$

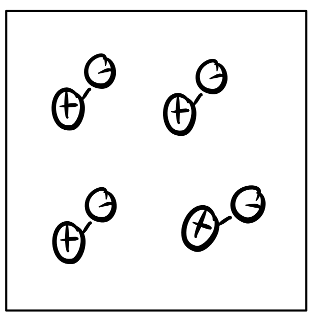
separation

Dipole!

$$\beta E_{\text{dipole}} = \frac{K}{2}\int d^2\underline{x}\ |\underline{\nabla}\theta|^2 \sim \pi K \log\left(\frac{l}{a}\right) + \underbrace{\int_l^\infty d^2\underline{x}\left(\frac{l}{x^2}\right)^2}_{\text{finite}}$$

- Deconfinement of vortices

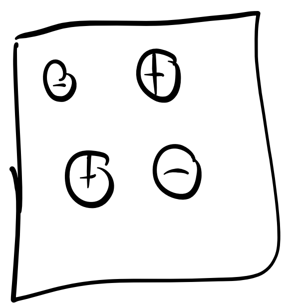
low T



dipole phase

$$\xrightarrow[\text{phase transit}]{T_c}$$

high T



plasma of unbound vortices

e.g. 2d melting
Superfluid films
Superconducting films

$n=3$ defect is known as a skyrmion
energy cost only finite