Kadanoff's Renormalisation Group

Gaussian Model Example

- This can be solved exactly – quadratic
  - Exercise for student!

$$\mathcal{Z} = \int \mathcal{D} m(q) \exp\left\{ -\int_0^{\Lambda \sim \frac{1}{a}} \frac{d^d q}{(2\pi)^d} \frac{1}{2} G_0^{-1}(q) |\underline{m}(q)|^2 + \underline{h} \cdot \underline{m}(q = \underline{0}) \right\},$$

where $G_0(q) = \dfrac{1}{t + Kq^2}$

We shall consider the phase transition from $t > 0$ side since we are discarding m^4 and higher terms for the moment.

Step 1: Coarse grain

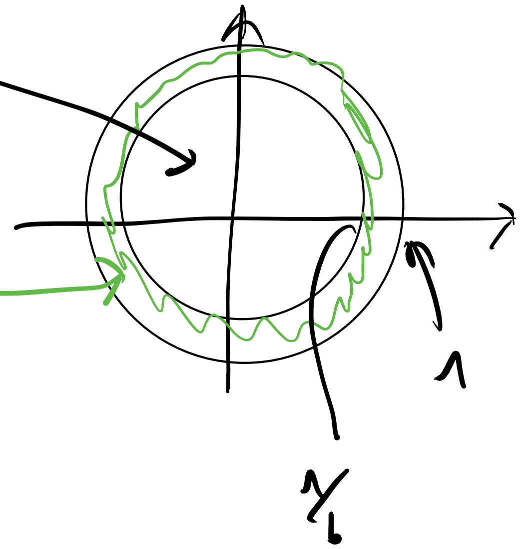
Eliminate $\begin{cases} \text{fluctuations at scales} \\ \text{modes} \end{cases}$

$$a < x < bx = ba$$

$$\frac{\Lambda}{b} < q < \Lambda \sim \frac{1}{a}$$

$m_<$ — slow

$m_>$ — fast



Modes decouple at Gaussian order

$$\beta H[m_>, m_<] = \beta H_0[m_<] + \beta H_0[m_>]$$

$$Z = \int \mathcal{D} m_>(q)\, e^{-\beta H_0[m_>]} \int \mathcal{D} m_<(q)\, e^{-\beta H_0[m_<]}$$

$$Z_> = \exp\left[-\frac{n}{2} V \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \log(t + K q^2)\right]$$

$$-V f_b(t) \leftarrow \text{analytic.}$$

$$\beta H_0[m_<] = \int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} \frac{1}{2} G_0^{-1} |m_<(q)|^2 - h \cdot m_<(q=0)$$

Step 2: Rescale

$$x' = \frac{x}{b} \quad \text{or} \quad q' = bq$$ – restores upper cut-off.

Step 3: Renormalize

$$\underline{m}'(\underline{x}') = \frac{1}{\zeta}\, \underline{m}_<(\underline{x}')$$

$$\underline{m}'(q') = \frac{1}{z}\, m_<(q')$$

Note ζ and z are different.

$$Z = Z_> \int \mathcal{D}\underline{m}'(q')\, e^{-\beta H'[m'(q')]}$$

$$\beta H' = \int_0^{\Lambda} \frac{d^d q'}{(2\pi)^d}\, b^{-d} z^2 \left(\frac{t + b^{-2} k q'^2}{2}\right) |\underline{m}'(q')|^2$$

$$- z\, \underline{h} \cdot \underline{m}'(q' = 0)$$

<u>Result</u> $S \equiv \{t, h, k\} \xrightarrow{RG} S' \equiv \{t', h', k'\}$

with
$$\begin{cases} K' = K\, b^{-d-2} z^2 \\ t' = t\, b^{-d} z^2 \\ h' = h z \end{cases}$$

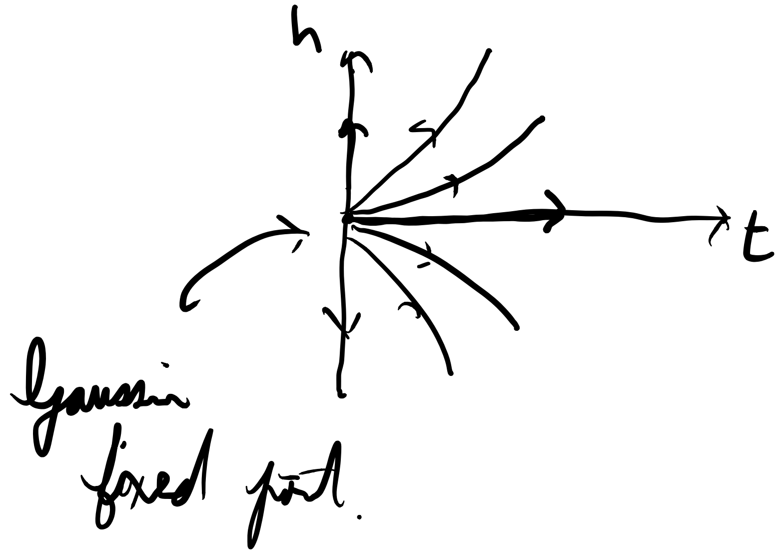
From physics considerations, these parameters are fixed at critical point

Fix K : $\quad z^2 = b^{d+2}$

$$\Rightarrow \quad z = b^{1+\frac{d}{2}}$$

$t^* = h^* = 0$ is a fixed point

$$\begin{cases} t' = b^2 t & \therefore\ y_t = 2 \\ h' = b^{1+\frac{d}{2}} h & \therefore\ y_h = 1+\frac{d}{2} \end{cases}$$



[for $t = t'$ fixed point.

K rescales to 0.

$\Rightarrow$ no coupling between spins / fluctuations.

$\rightsquigarrow$ high T phase.]

$$\Rightarrow \quad f_{sing}(t,h) = b^{-d} f_{sing}(b^2 t, b^{1+\frac{d}{2}} h)$$

$$= t^{\frac{d}{2}} g_f\left(\frac{h}{t^{\frac{1}{2}+\frac{d}{4}}}\right) \qquad b^2 t = 1$$

i.e. we have justified our homogeneity assumption!

Fixed Hamiltonian $(t^* = h^* = 0)$

$$\beta H^* = \frac{1}{2} K \int d^d \underline{x} \, |\nabla \underline{m}|^2$$

Use scale invariance to fix ζ

$$\underline{x} = b\underline{x}' \qquad \underline{m}(\underline{x}') = \zeta \, m'(\underline{x}')$$

$$\beta H^* = \frac{1}{2} K \zeta^2 b^{d-2} \int d^d \underline{x}' \, |\nabla \underline{m}'|^2$$

$$\Rightarrow \quad \zeta = b^{1-\frac{d}{2}}$$

Stability of fixed point

$$\beta H = \beta H^* + u_p \int d^d\underline{x}\; m^p$$

$$\longrightarrow \beta H^* + u_p\, b^d \zeta^p \int d^d\underline{x}' (m')^p$$

$$\Rightarrow \quad u_p \rightarrow u_p' = b^d\, b^{p-\frac{pd}{2}}\, u_p$$

$$= b^{p-d\left(\frac{p}{2}-1\right)}\, u_p$$

$$= b^{y_p}$$

$$y_p = p - d\left(\frac{p}{2}-1\right)$$

e.g. $\quad y_1 \equiv y_h = 1 + \frac{d}{2}$

$$y_2 \equiv y_t = 2$$

$$y_4 = 4 - d$$

$$u_4 \begin{cases} \text{relevant (grows) for } d<4 \\ \text{irrelevant (shrinks) for } d>4 \end{cases}$$

In $d>4$ small m^4 perturbation have no effect at fixed point

Recall : 4 is the upper critical dimension for M.F.T.

$y_6 = 6-2d$; u_6 is relevant for $d<3$

<u>Generally</u>

If $\underline{S} \equiv (t, u, \dots, k, \dots)$ and

$$\underline{S} = \underline{S}^* + \delta \underline{S}$$
$$\downarrow \text{R.G.}$$
$$\underline{S}' = \underline{S}^* + R\, \delta \underline{S}$$

$\uparrow$ stability matrix

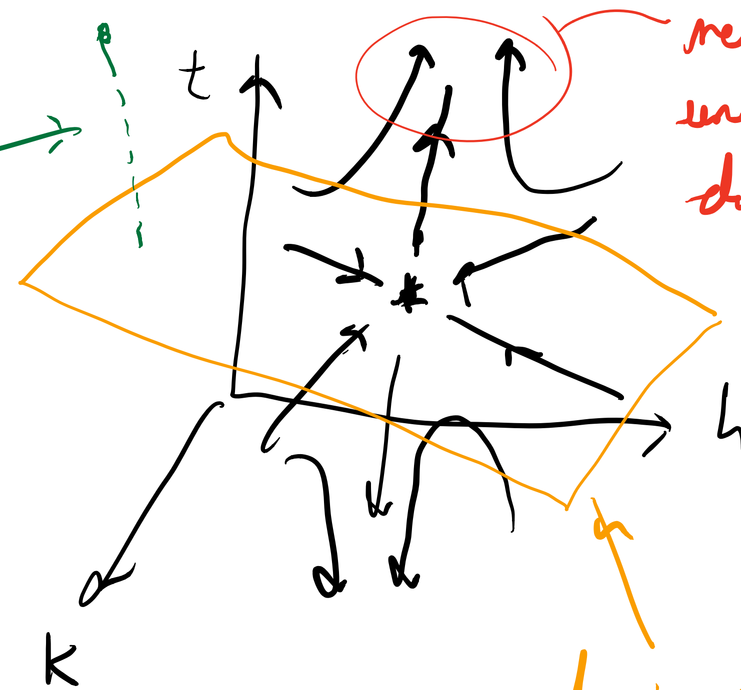
with eigenvalues b^{y_α}

eigenvectors θ_α

$$\beta H = \beta H^* + \sum_\alpha{}' g_\alpha \theta_\alpha$$

$$\downarrow$$

$$\beta H' = \beta H^* + \sum_\alpha{}' g_\alpha b^{y_\alpha} \theta_\alpha$$



relevant unstable direction $y_\alpha > 0$ e.g. h, t.

experimental trajectory to a phase transition

basin of attraction

stable, irrelevant direction

$y_\alpha < 0$ e.g. u in $d > 4$.

[If $y_\alpha = 0$, direction is marginal

– requires higher order calculation.]