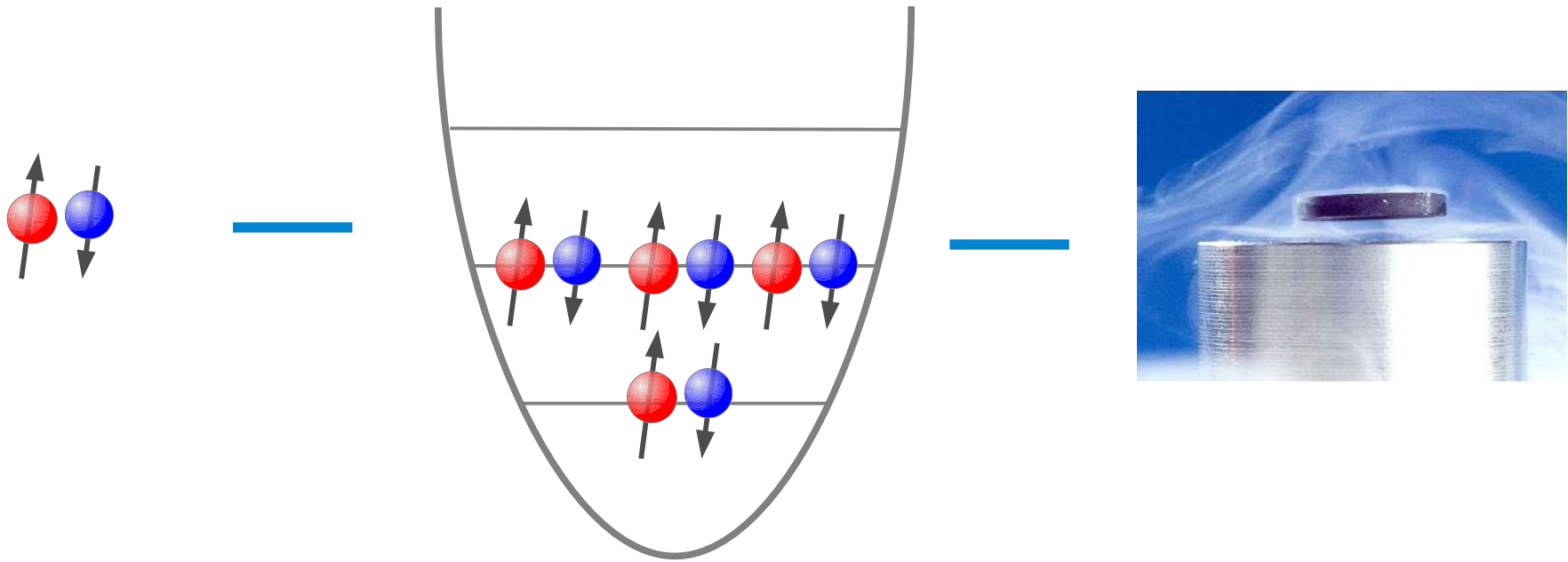


# A few-fermion BCS superconductor

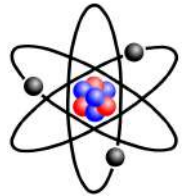
**J. Lofthouse & G.J. Conduit**

Theory of Condensed Matter Group, Department of Physics, Cambridge

# Superconductivity and Scale in Quantum Systems

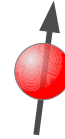


# Experimental Realization of a Cold Atom Gas



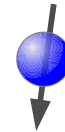
${}^6\text{Li}$  atom

$$|F = 1/2, m_F = 1/2\rangle$$

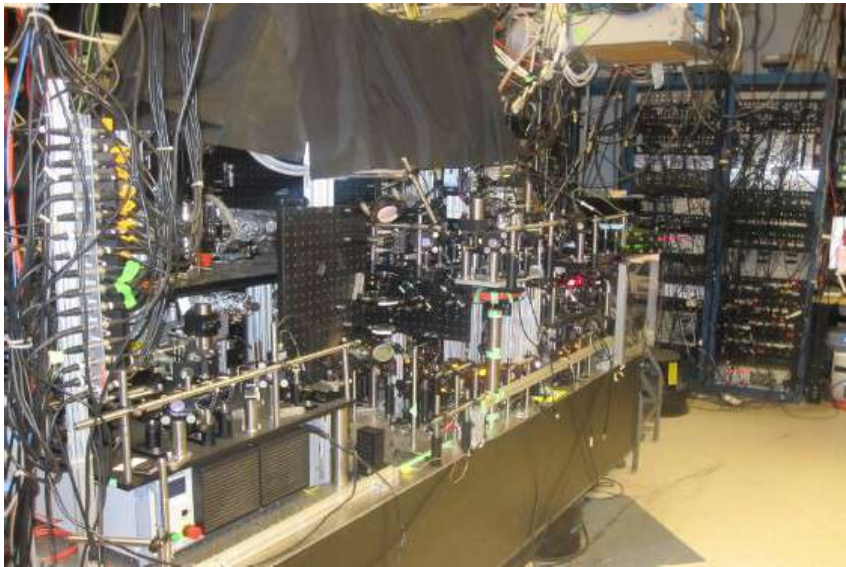


Up spin electron

$$|F = 1/2, m_F = -1/2\rangle$$



Down spin electron

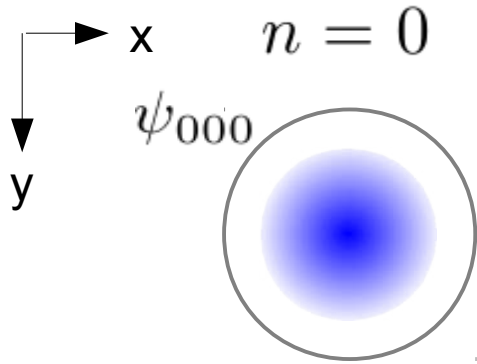


# A Model of a Cold Atom Gas

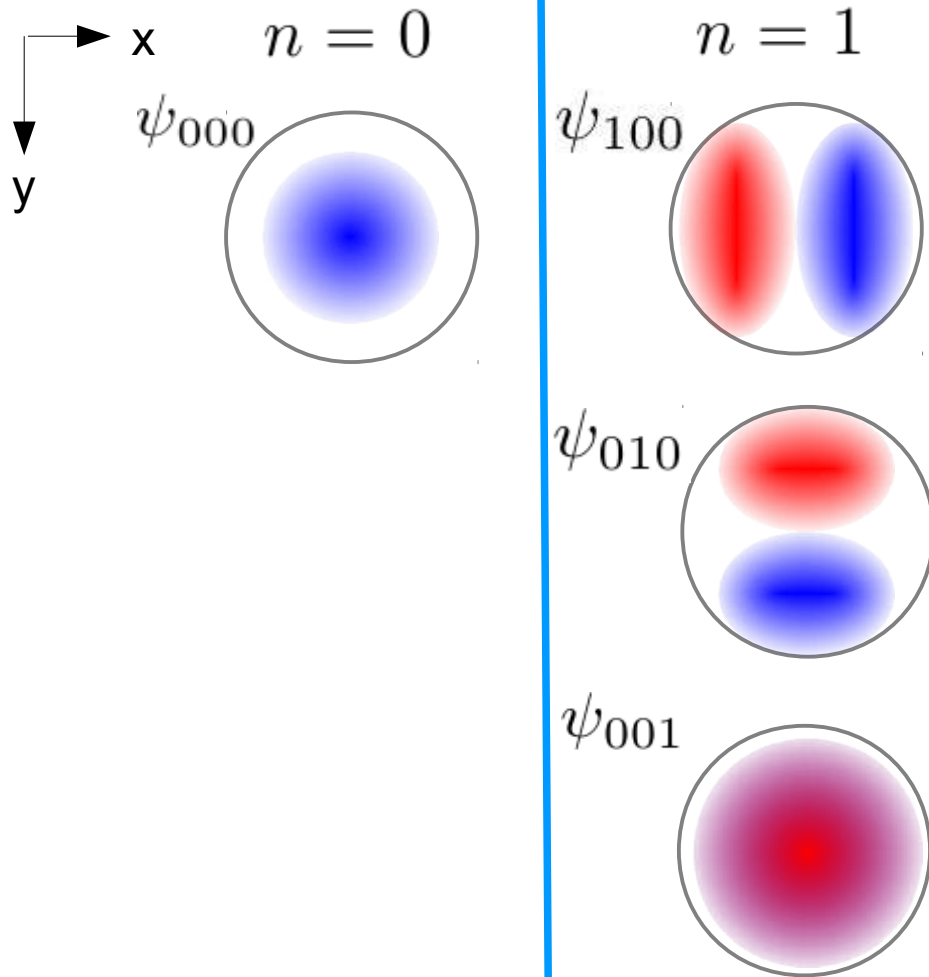
- Optical trap modeled by 3-D QHO

$$\hat{H}^{(0)} = \frac{-\hbar^2}{2m} \nabla^2 + \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$$

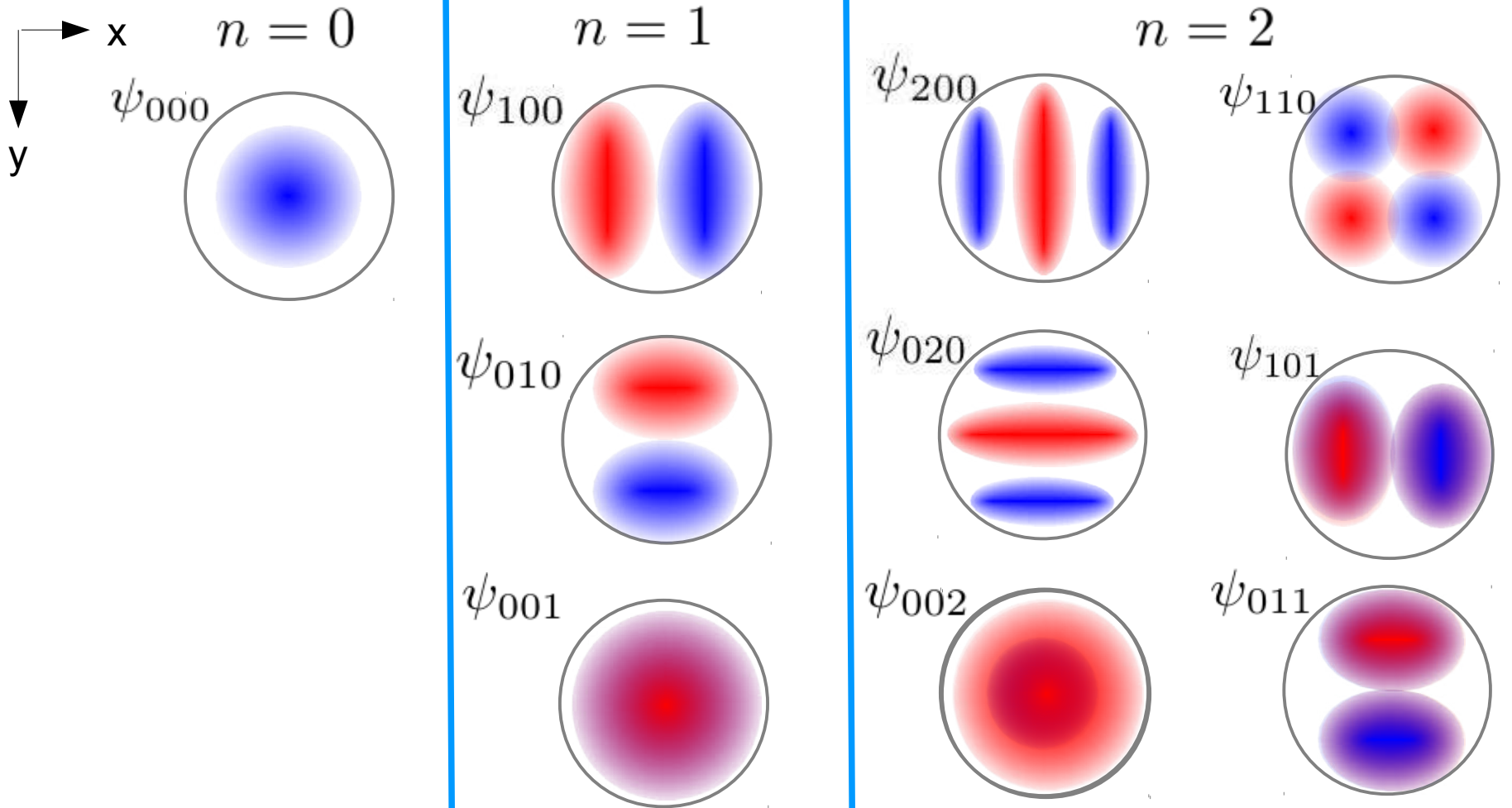
# 3-D QHO Orbitals



# 3-D QHO Orbitals



# 3-D QHO Orbitals



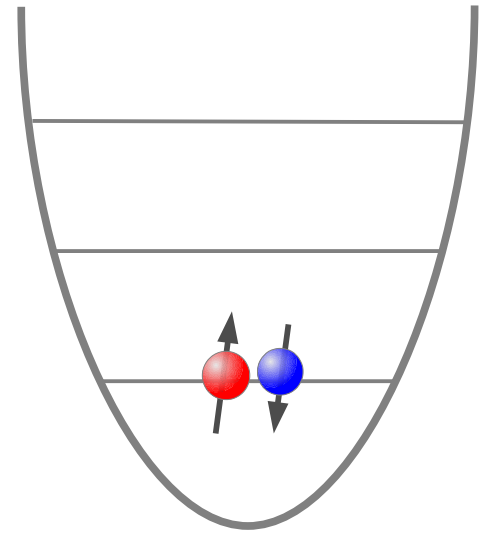
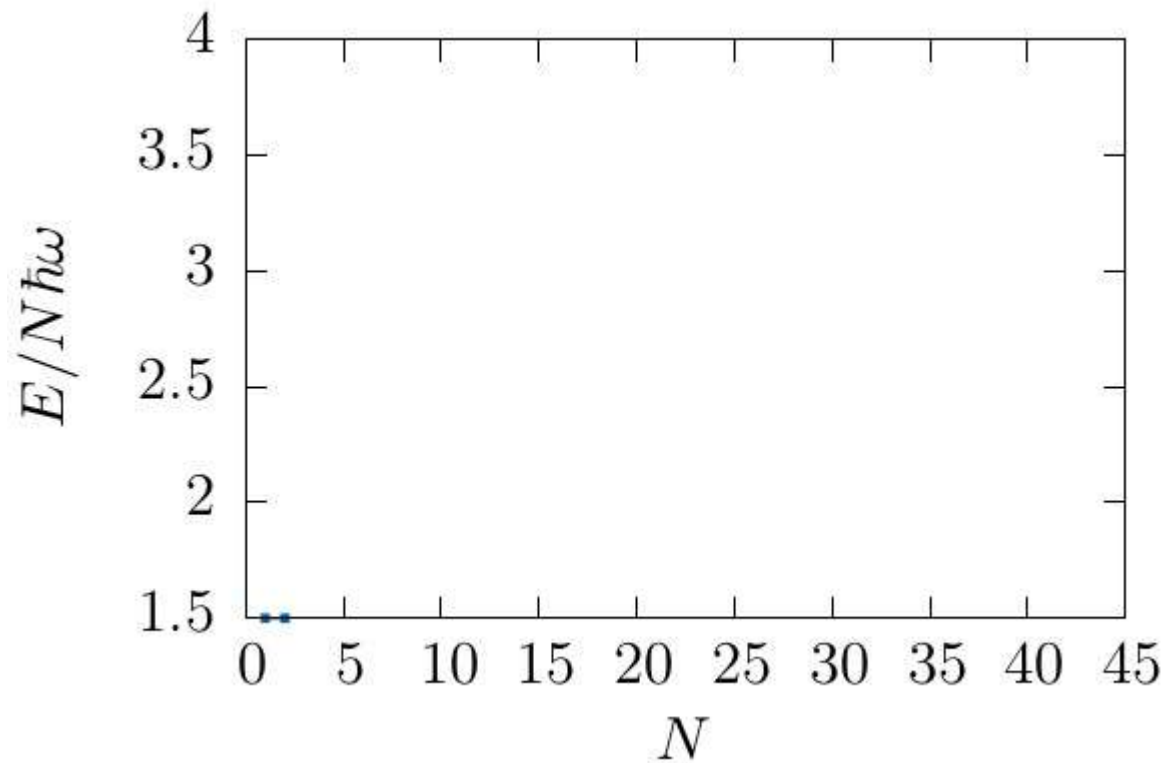
## 3-D QHO Orbitals

$$E = \hbar\omega (n + 3/2)$$

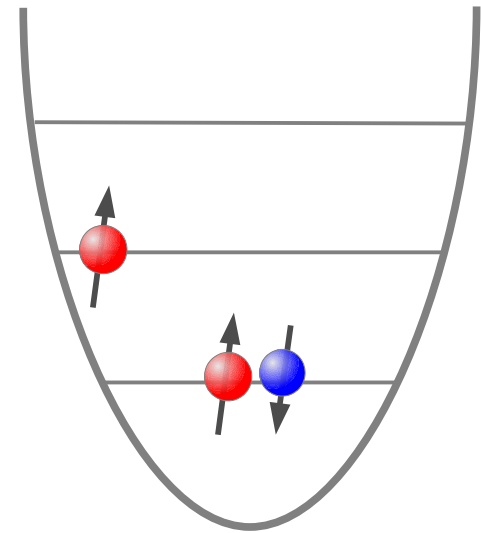
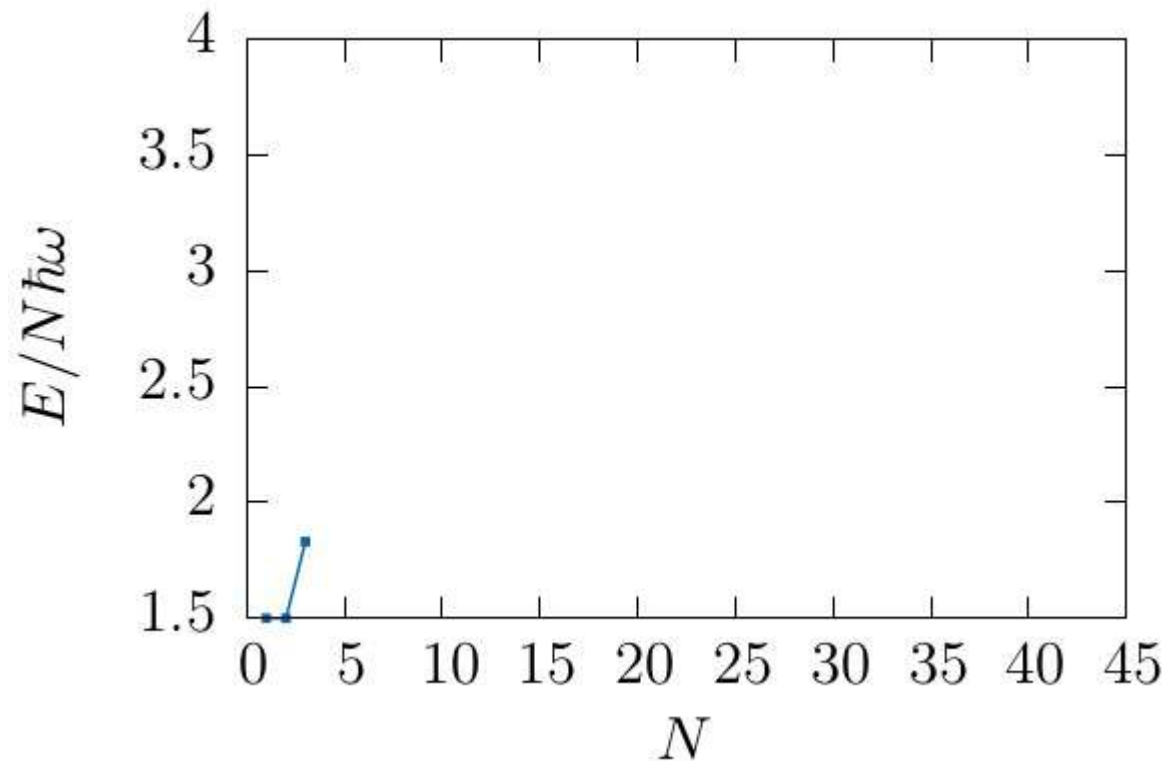
$$n = n_x + n_y + n_z$$



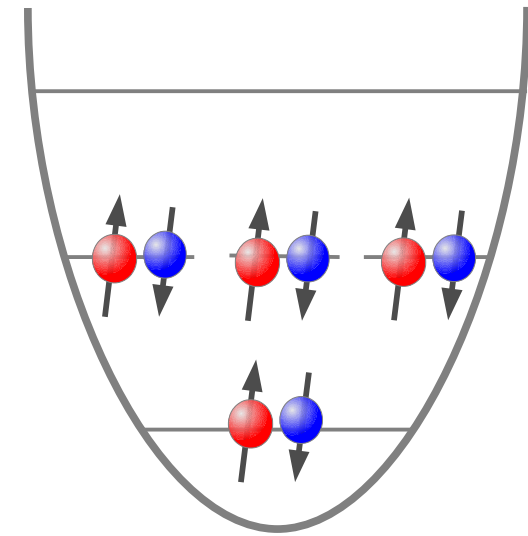
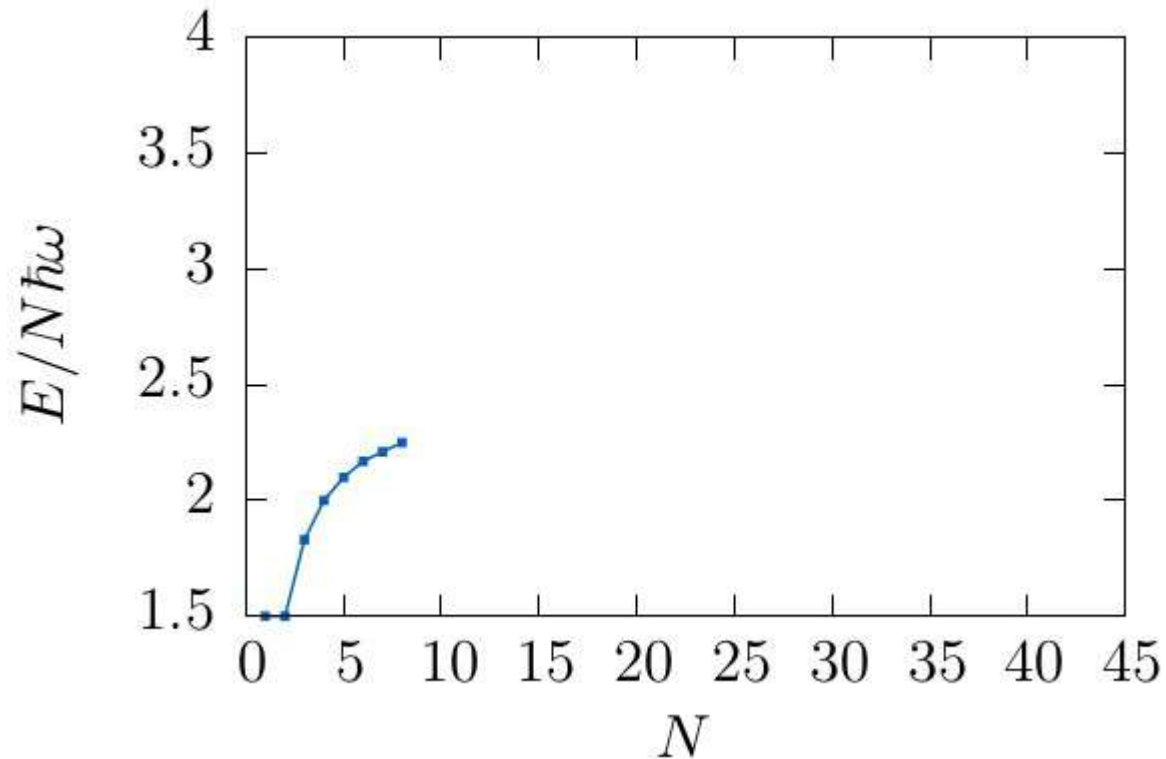
# Non-Interacting System Ground State



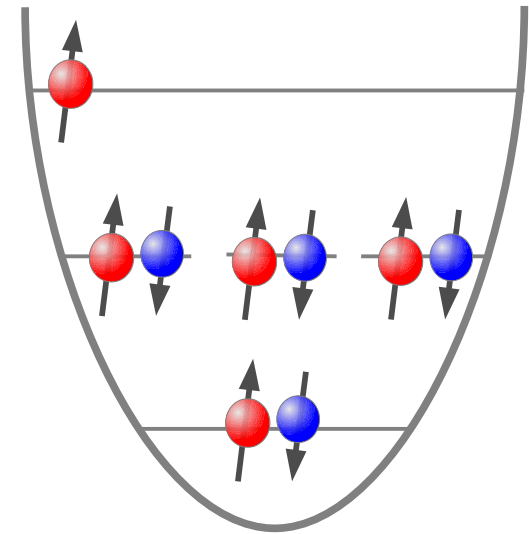
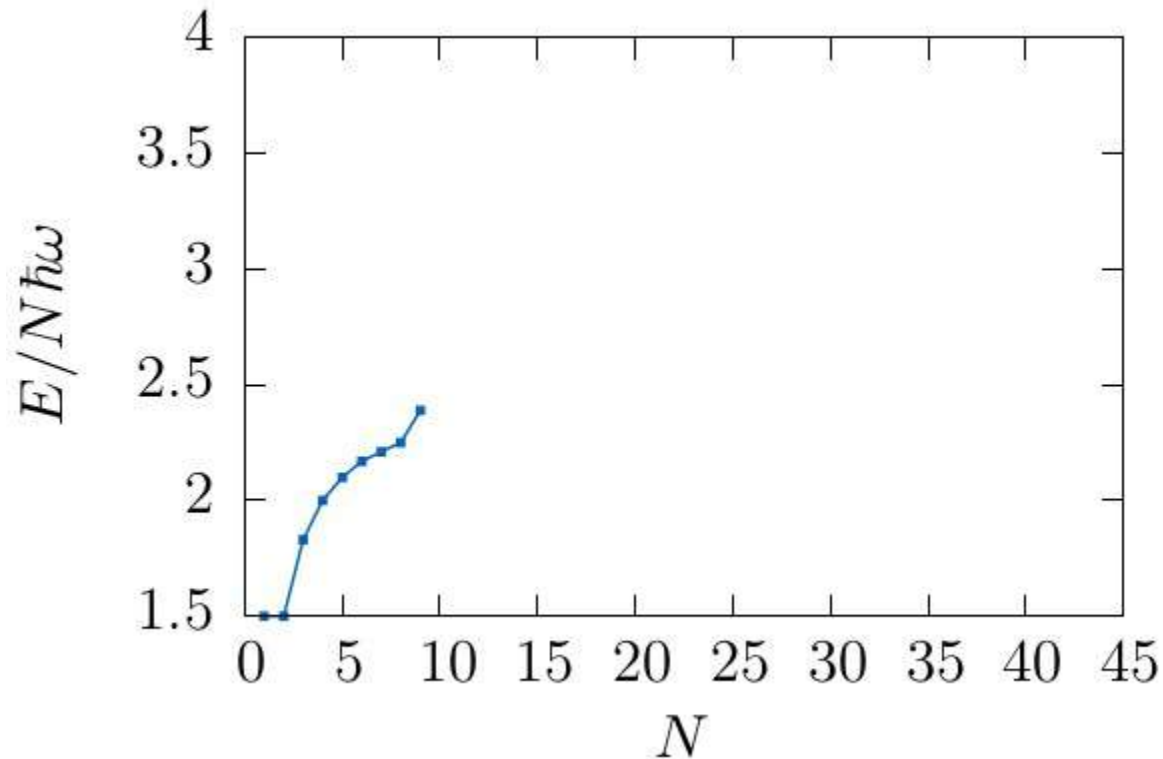
# Non-Interacting System Ground State



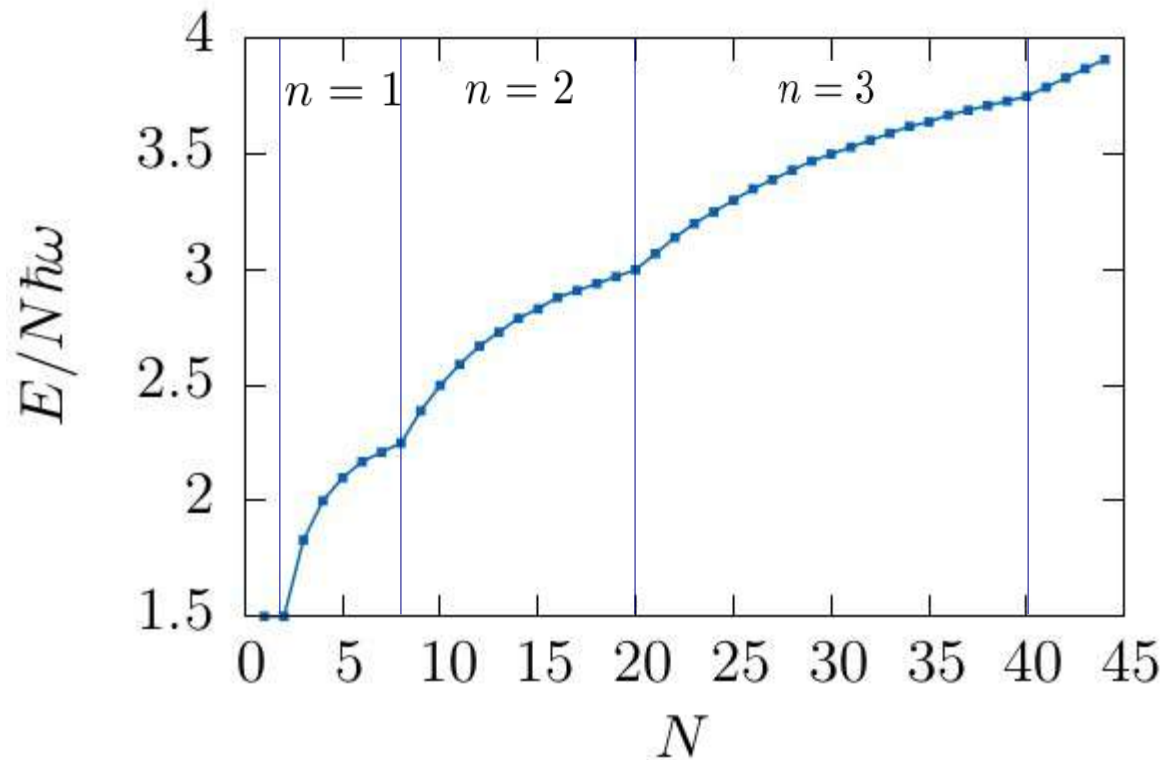
# Non-Interacting System Ground State



# Non-Interacting System Ground State



# Non-Interacting System Ground State



# Quantum Monte-Carlo

- Non-interacting system orbitals suitable basis for weak interactions

$$\psi_{n_x n_y n_z}(\mathbf{r}) = H_{n_x}(\sqrt{\omega_x}x) e^{-\frac{1}{2}(\sqrt{\omega_x}x)^2} \times H_{n_y}(\sqrt{\omega_y}y) e^{-\frac{1}{2}(\sqrt{\omega_y}y)^2} \times H_{n_z}(\sqrt{\omega_z}z) e^{-\frac{1}{2}(\sqrt{\omega_z}z)^2}$$

- G.S. Energy Monte-Carlo integration

$$E^{\text{est}} = \frac{\int d^{3N} \mathbf{r} \Psi_{\text{trial}}^*(\mathbf{r}_1, \dots, \mathbf{r}_N) \hat{H} \Psi_{\text{trial}}(\mathbf{r}_1, \dots, \mathbf{r}_N)}{\int d^{3N} \mathbf{r} \Psi_{\text{trial}}^*(\mathbf{r}_1, \dots, \mathbf{r}_N) \Psi_{\text{trial}}(\mathbf{r}_1, \dots, \mathbf{r}_N)}$$

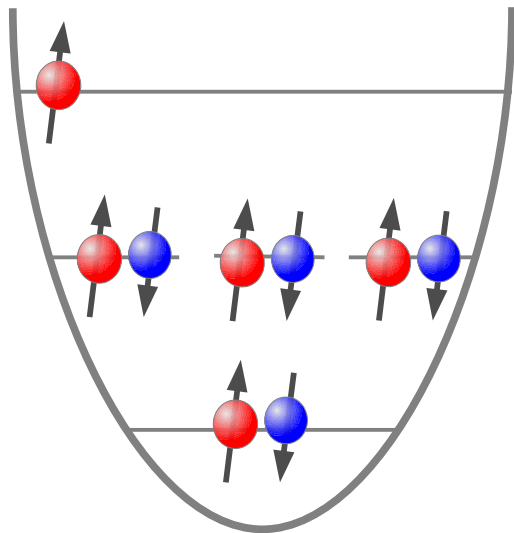
# Superfluid Hamiltonian in External Trap

- Pairwise attractive interactions between up and down spins

$$\hat{H} = \hat{H}^{(0)} + \iint d^3\mathbf{r}d^3\mathbf{r}' c_{\uparrow}^{\dagger}(\mathbf{r})c_{\downarrow}^{\dagger}(\mathbf{r}')V(\mathbf{r} - \mathbf{r}')c_{\downarrow}(\mathbf{r}')c_{\uparrow}(\mathbf{r})$$

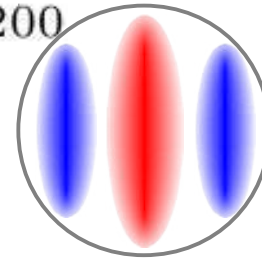
$$V(\mathbf{r} - \mathbf{r}') = -V_0\delta(\mathbf{r} - \mathbf{r}')$$

# Effect of Interactions on Degeneracy

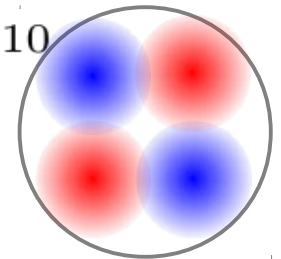


$n = 2$

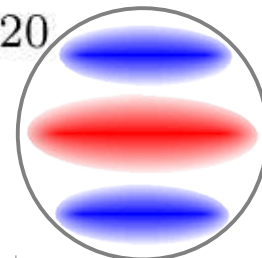
$\psi_{200}$



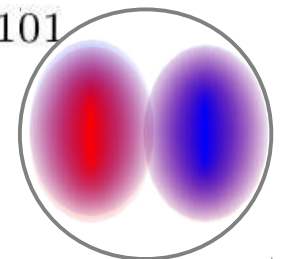
$\psi_{110}$



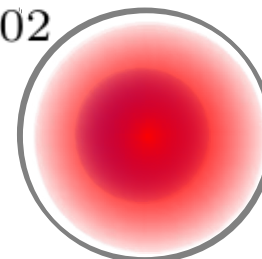
$\psi_{020}$



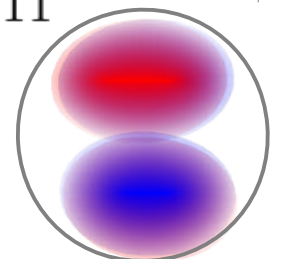
$\psi_{101}$



$\psi_{002}$



$\psi_{011}$

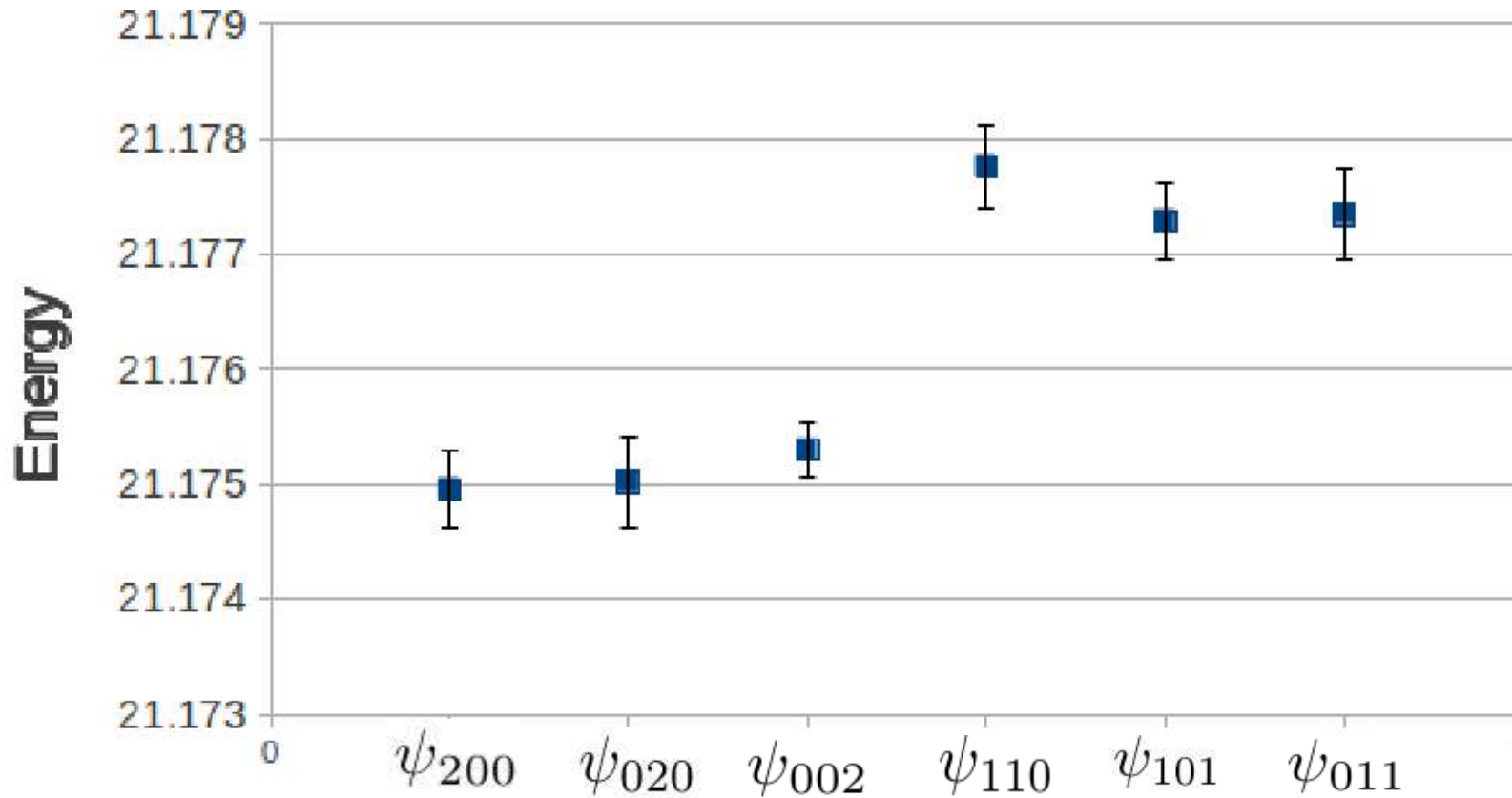


$$D_{\uparrow}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5) =$$

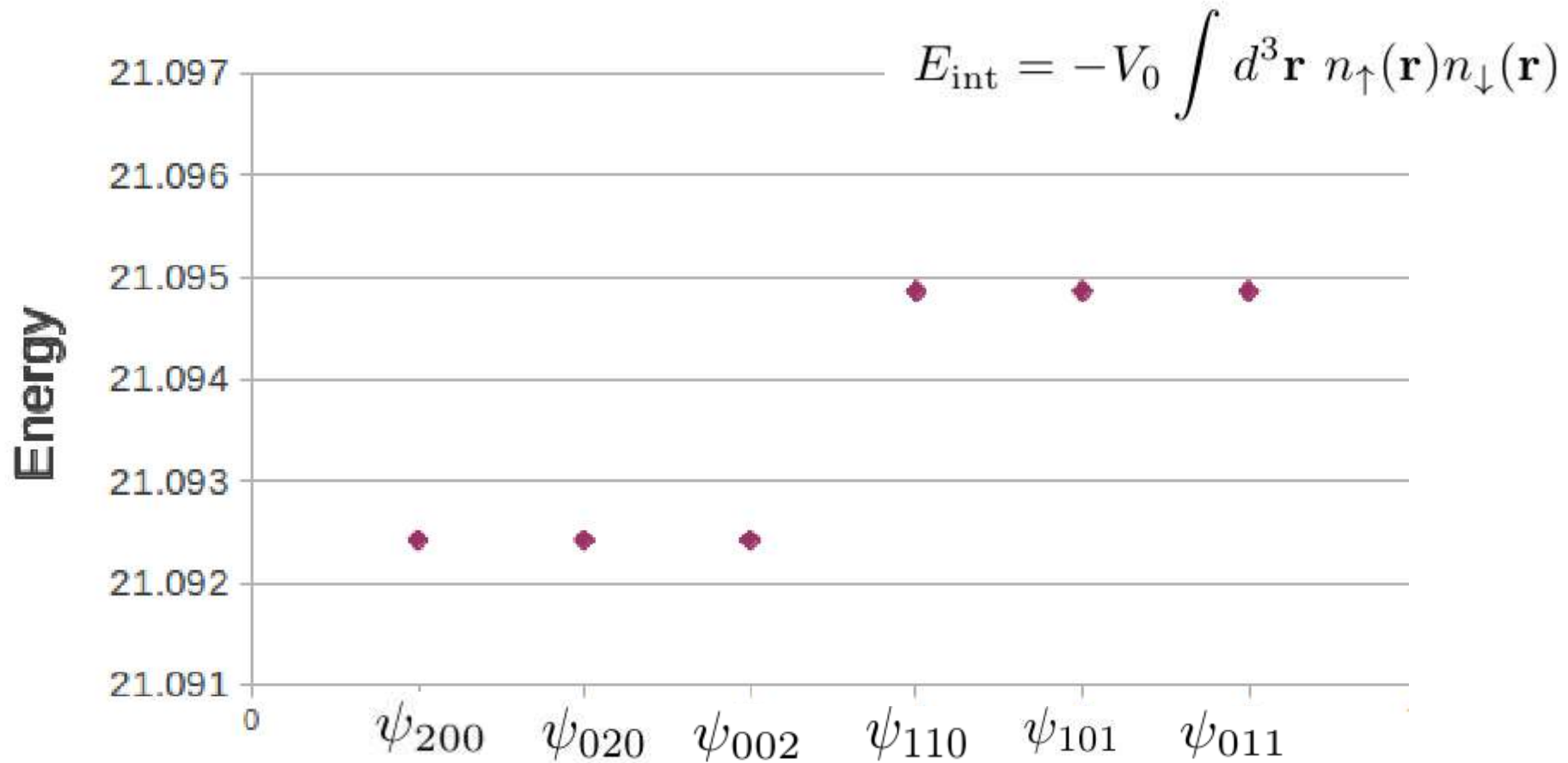
$$\mathcal{A} \prod \psi_{000}(\mathbf{r}_1) \psi_{100}(\mathbf{r}_2) \psi_{010}(\mathbf{r}_3) \psi_{001}(\mathbf{r}_4) \psi_j(\mathbf{r}_5)$$



# Effect of Interactions on Degeneracy



# Effect of Interactions on Degeneracy



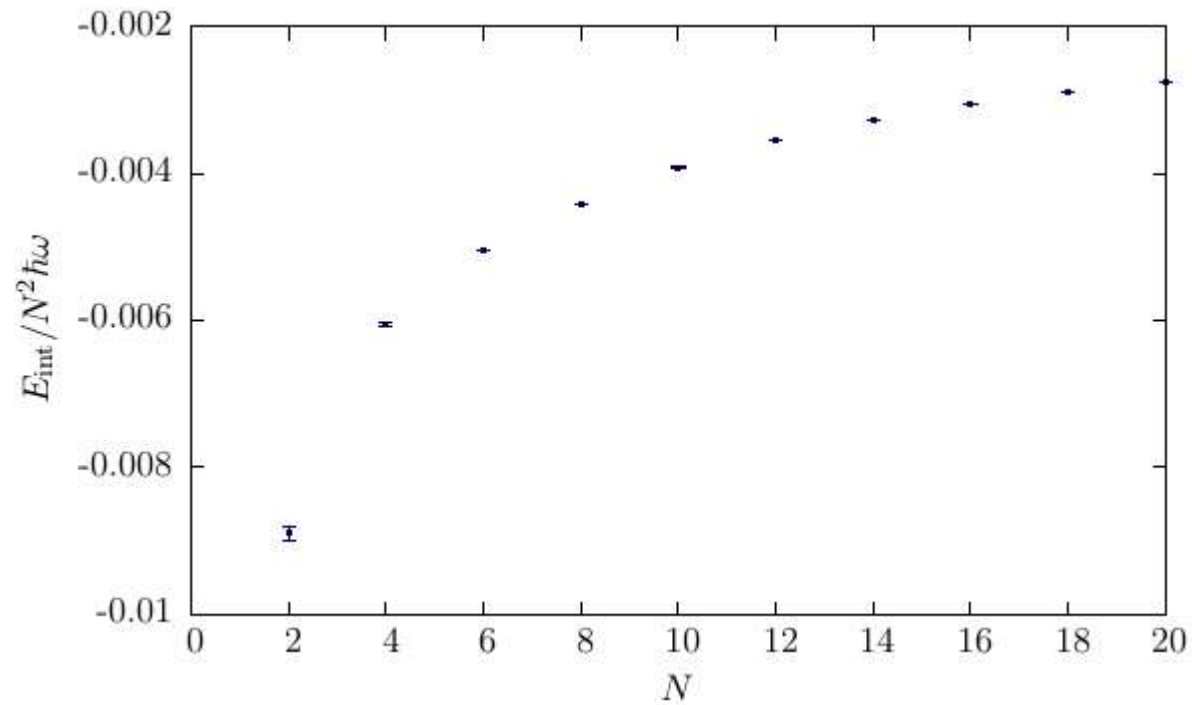
# Interaction Energy

$$E_{\text{int}} = -V_0 \left\langle \int d^3 \mathbf{r} c_{\uparrow}^{\dagger} c_{\uparrow} c_{\downarrow}^{\dagger} c_{\downarrow} \right\rangle$$
$$\rightarrow -V_0 \int d^3 \mathbf{r} n_{\uparrow} n_{\downarrow}$$

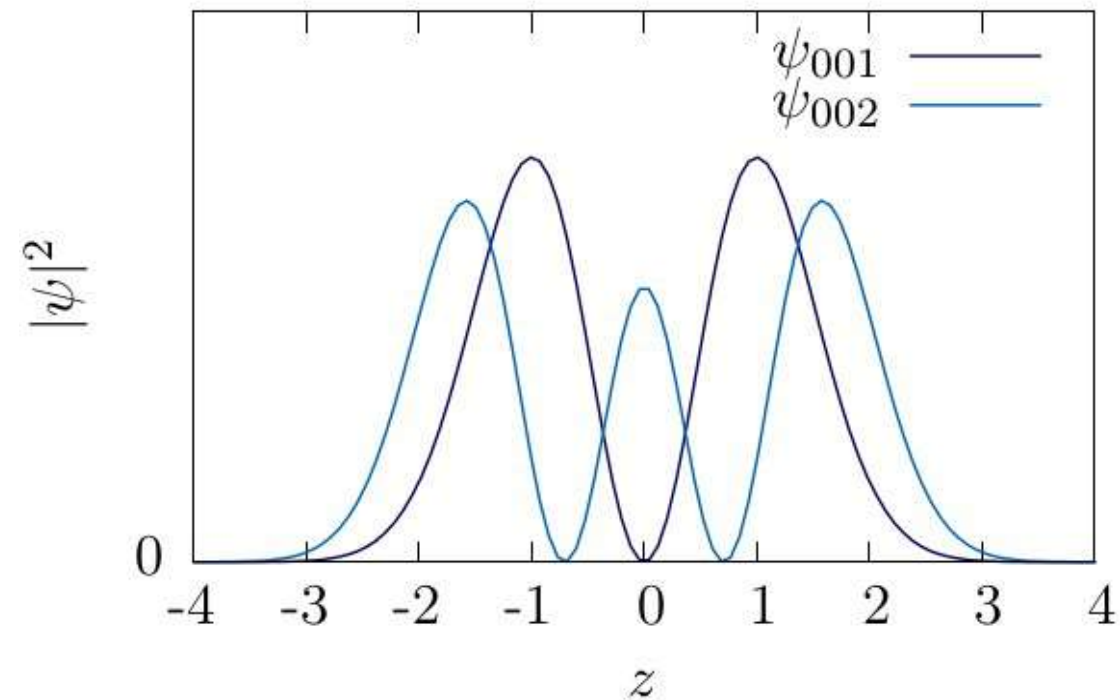
# Interaction Energy

$$\begin{aligned} E_{\text{int}} &= -V_0 \left\langle \int d^3\mathbf{r} c_{\uparrow}^{\dagger} c_{\uparrow} c_{\downarrow}^{\dagger} c_{\downarrow} \right\rangle \\ &\rightarrow -V_0 \int d^3\mathbf{r} n_{\uparrow} n_{\downarrow} \\ &\simeq -V_0 \frac{N_{\uparrow} N_{\downarrow}}{L^3} \\ &= -V_0 \frac{N^2}{4L^3} \end{aligned}$$

# Interaction Energy

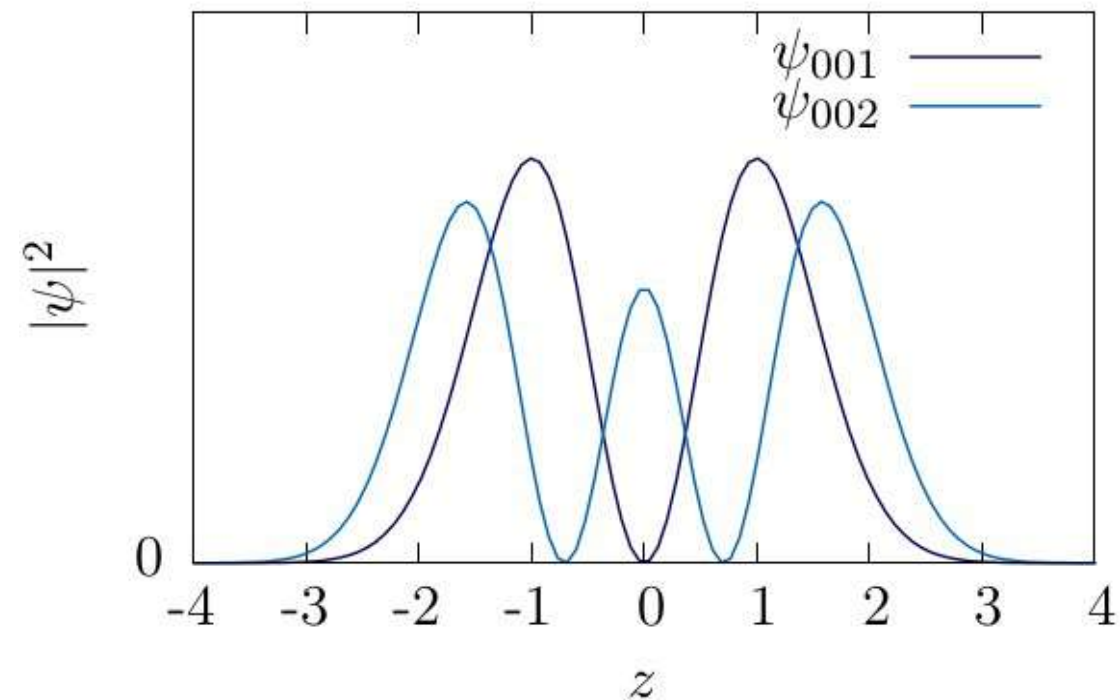


# System Geometry



$$\frac{E_{\text{int}}}{N^2} = -\frac{V_0}{4L^3}$$

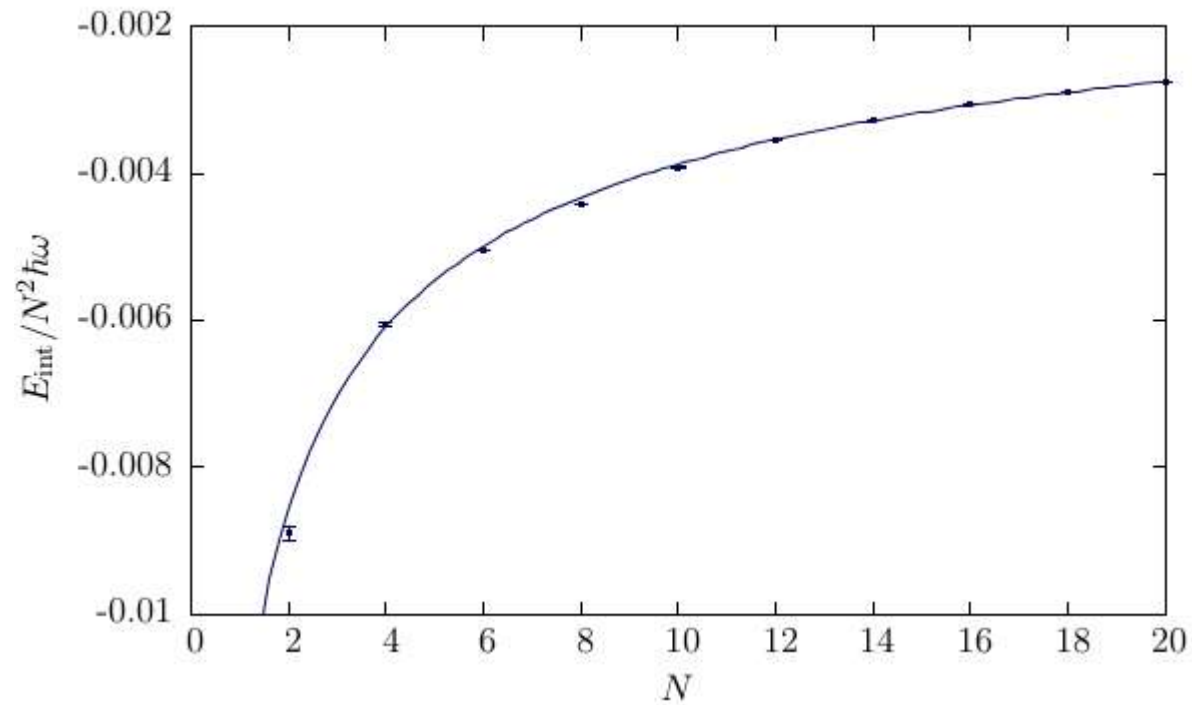
# System Geometry



$$\frac{E_{\text{int}}}{N^2} = -\frac{V_0}{4L^3}$$
$$L^3 \sim N^{1/2}$$

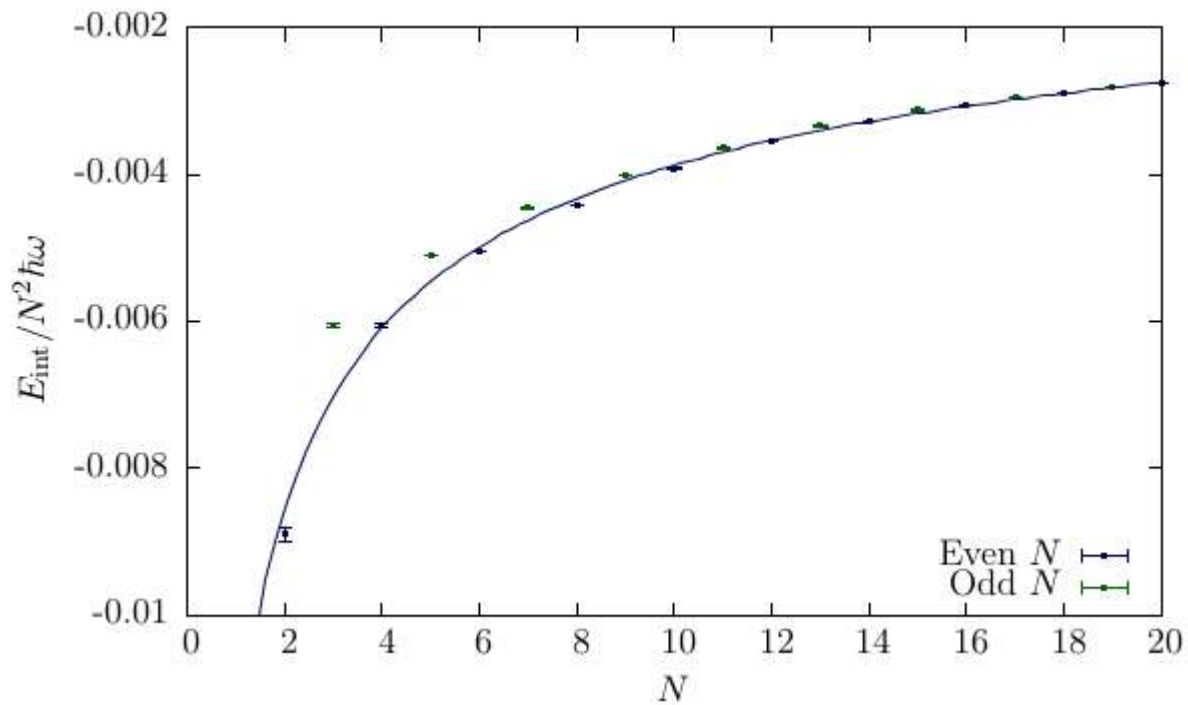
$$\frac{E_{\text{int}}}{N^2} \sim -V_0 N^{-1/2}$$

# System Geometry



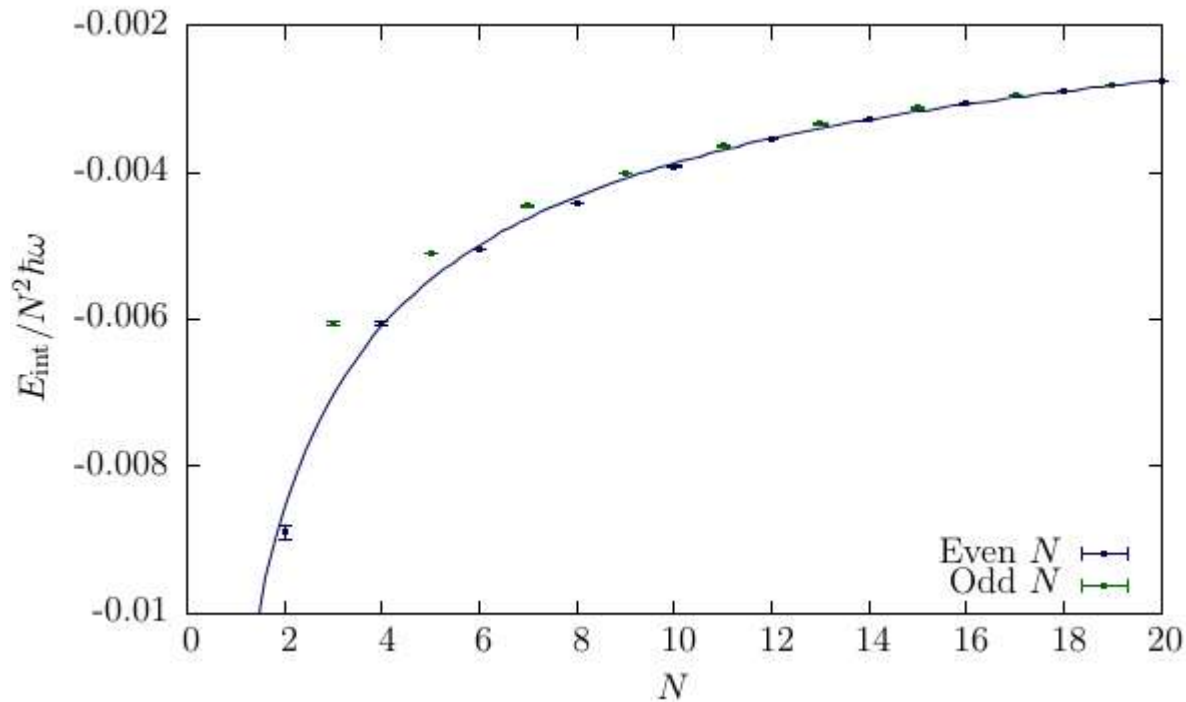


# Pairing Effect



# Pairing Effect

$$\frac{E_{\text{int}}}{N^2 \hbar \omega} \sim -\frac{V_0}{N^{\frac{1}{2}}} \left( 1 - 4 \frac{M^2}{N^2} \right)$$



# Asymmetric Trap

$$\hat{H}^{(0)} = \frac{-\hbar^2}{2m} \nabla^2 + \frac{1}{2} m (\omega_{\perp}^2 x^2 + \omega_{\perp}^2 y^2 + \omega_{\parallel}^2 z^2)$$

Non-interacting energy of  $N = 8$  state

$$E = 6\hbar(2\omega_{\perp} + \omega_{\parallel})$$

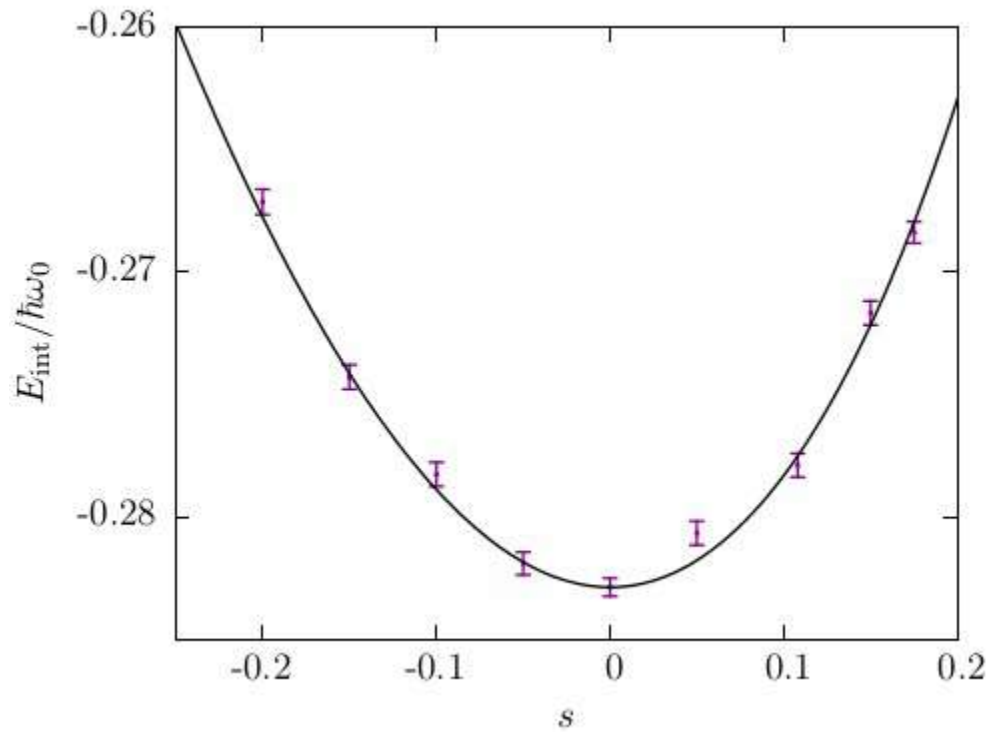
Kept constant for

$$\omega_{\parallel}(s) = \omega_0(1 - 2s)$$

$$\omega_{\perp}(s) = \omega_0(1 + s)$$

# Asymmetric Trap

$$\frac{E_{\text{int}}}{\hbar\omega_0} \simeq -V_0\omega_0^{\frac{1}{2}} \frac{N^2}{\hbar^{\frac{3}{2}}} (1+s)\sqrt{1-2s}$$

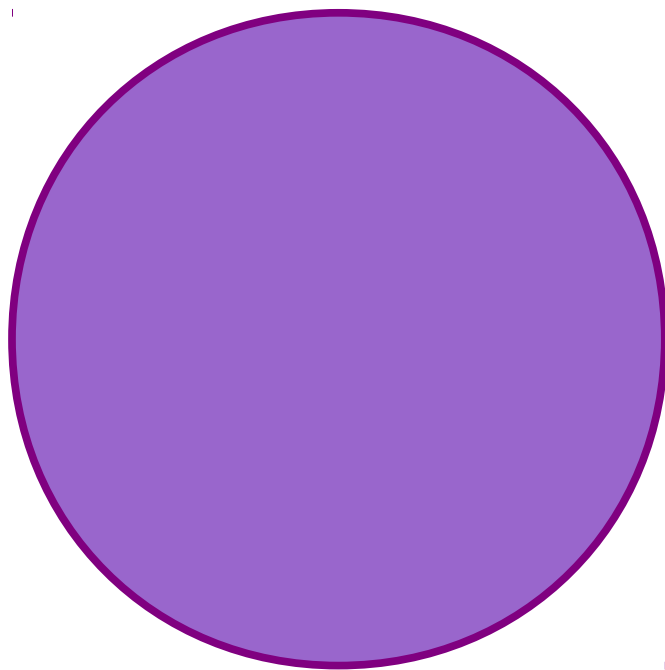


$$L^3 \simeq \sqrt{\langle \hat{x}^2 \rangle \langle \hat{y}^2 \rangle \langle \hat{z}^2 \rangle} \sim \frac{1}{\omega_{\perp} \sqrt{\omega_{\parallel}}}$$

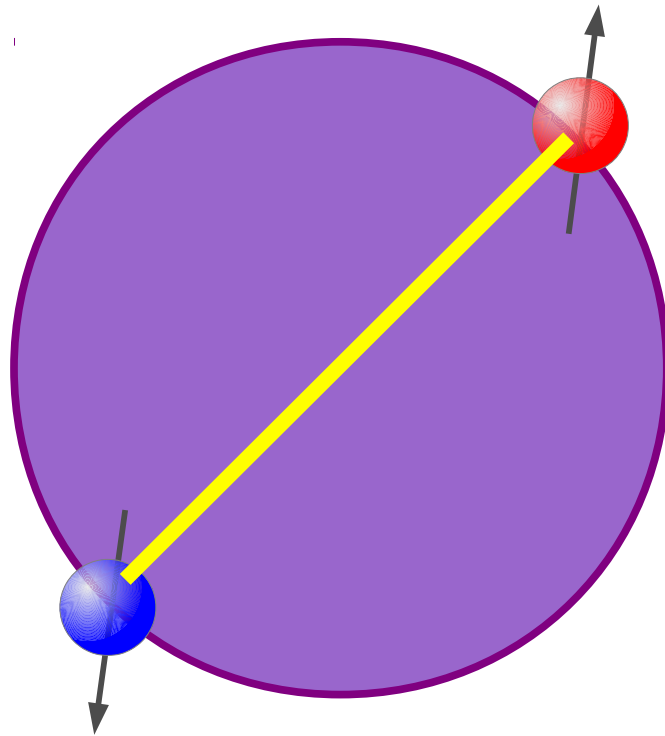
# Summary

- Unique opportunity to realize a few body interacting system
- Link between microscopic physics and macroscopic phenomenology
- DMC simulations allowed us to probe attractive interactions

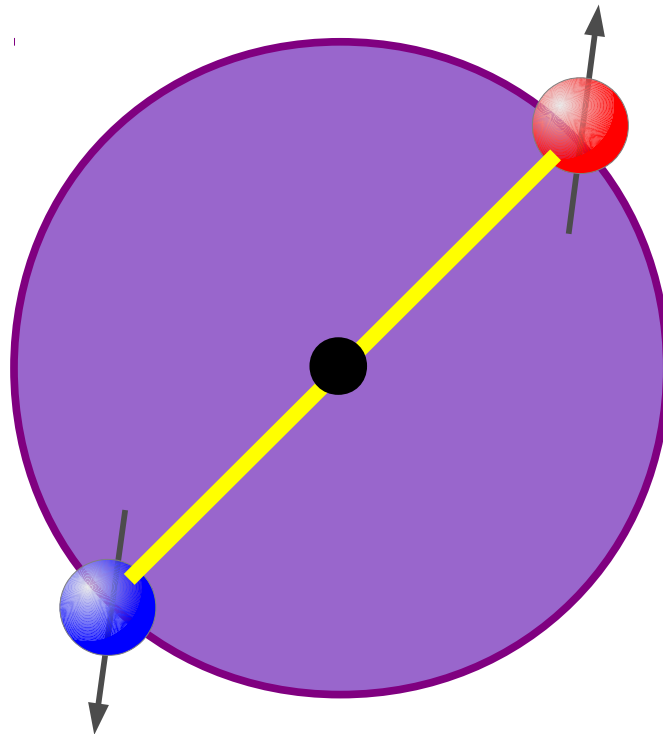
# BCS state



# BCS state

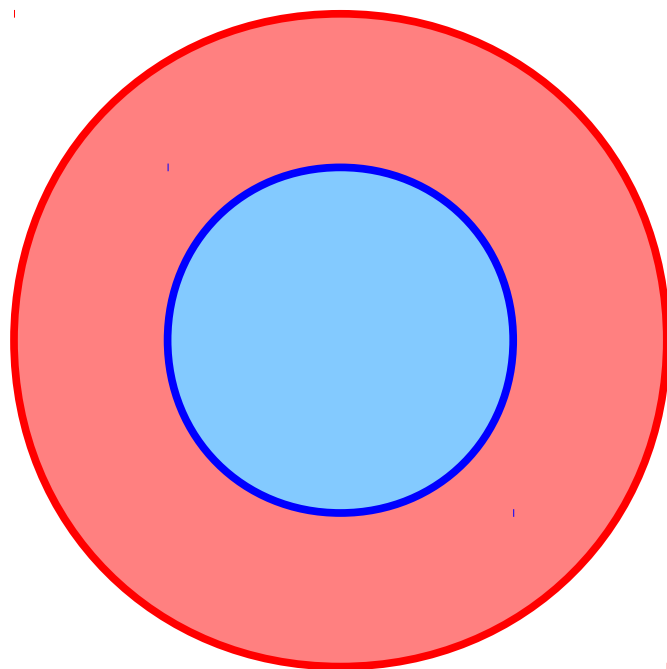


# BCS state

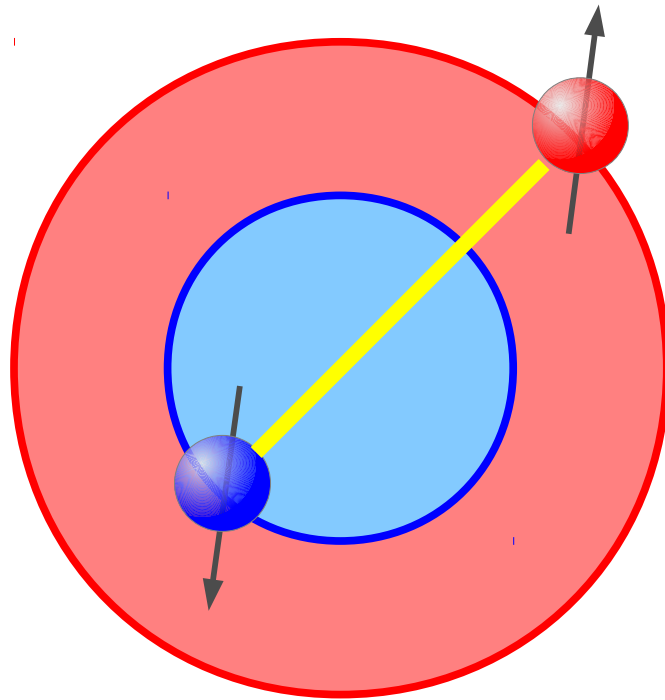




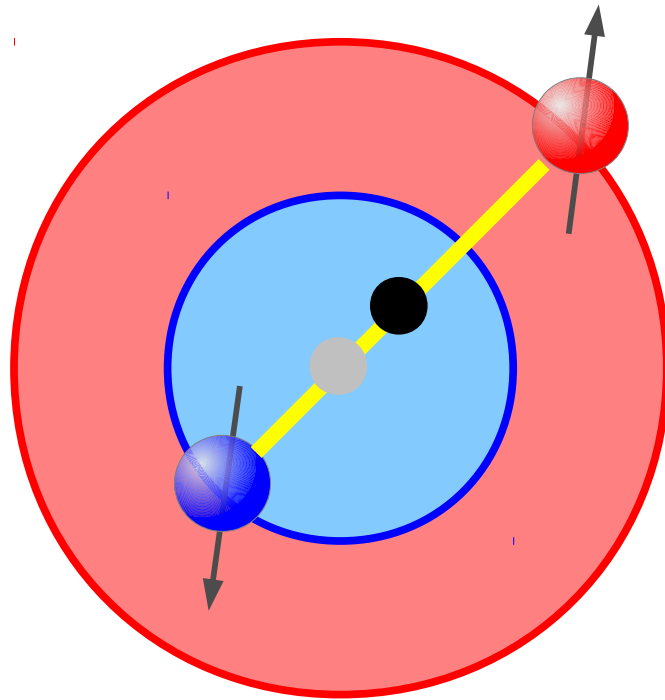
# BCS FFLO state



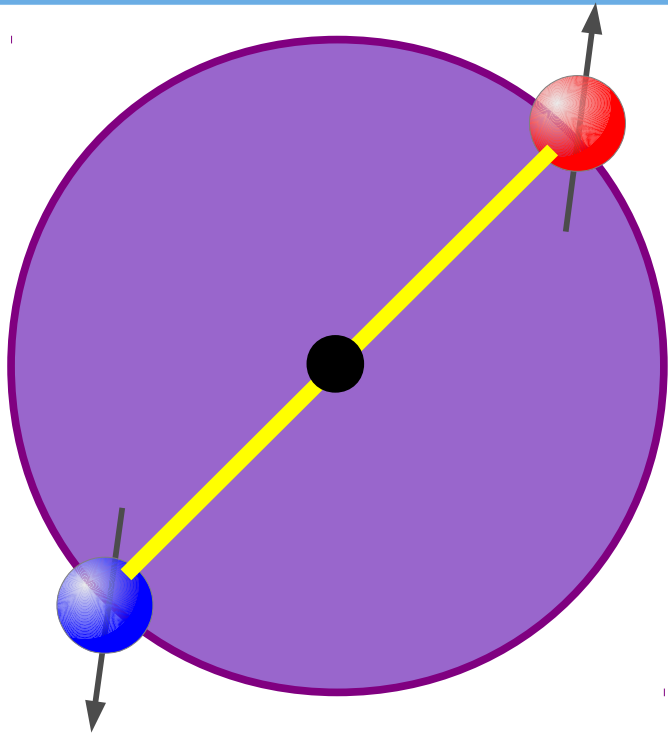
# BCS FFLO state



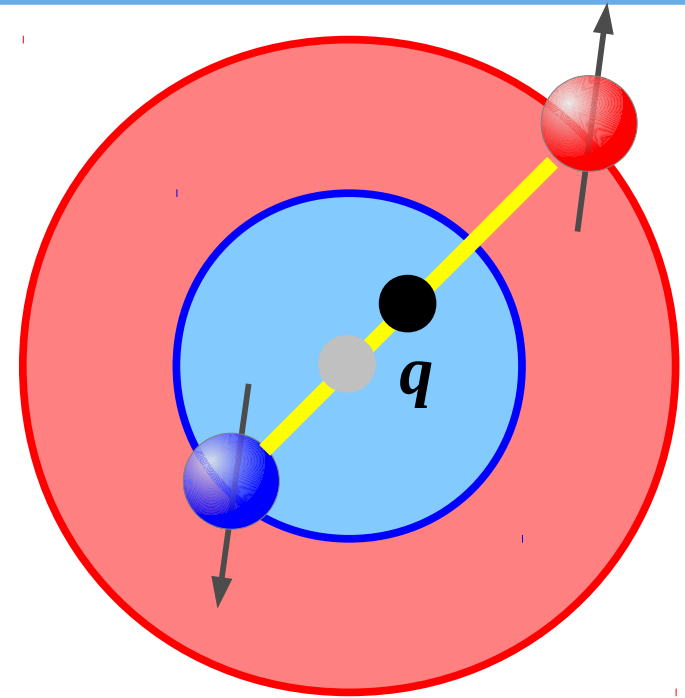
# BCS FFLO state



# BCS FFLO state

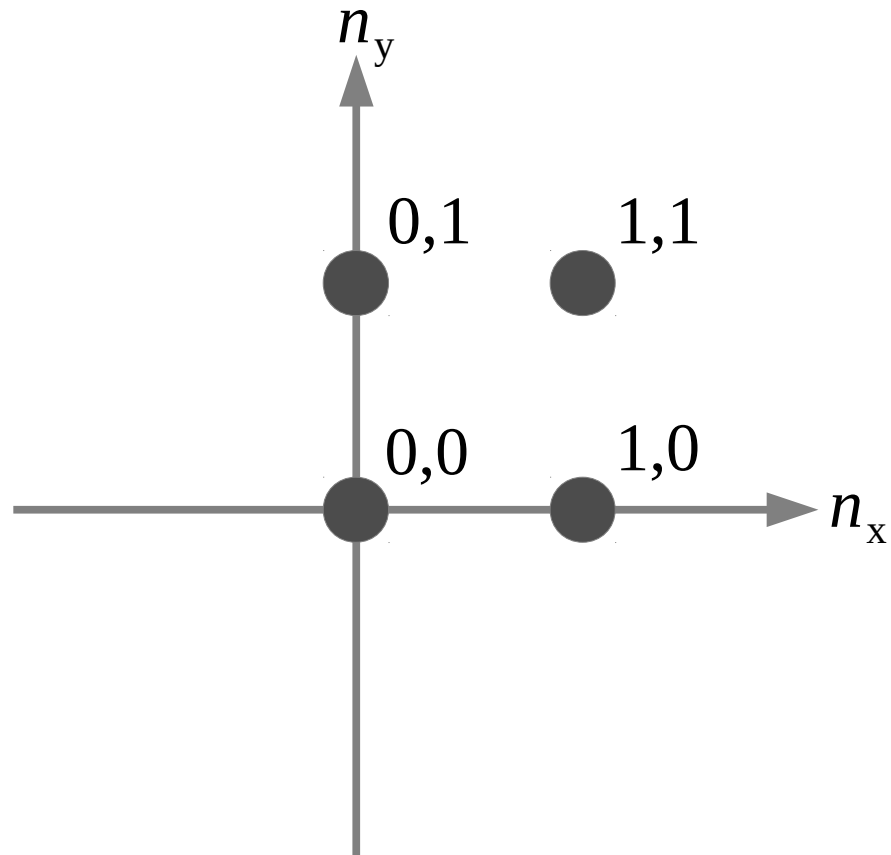


$$\Delta(\mathbf{r}) = \Delta_0$$

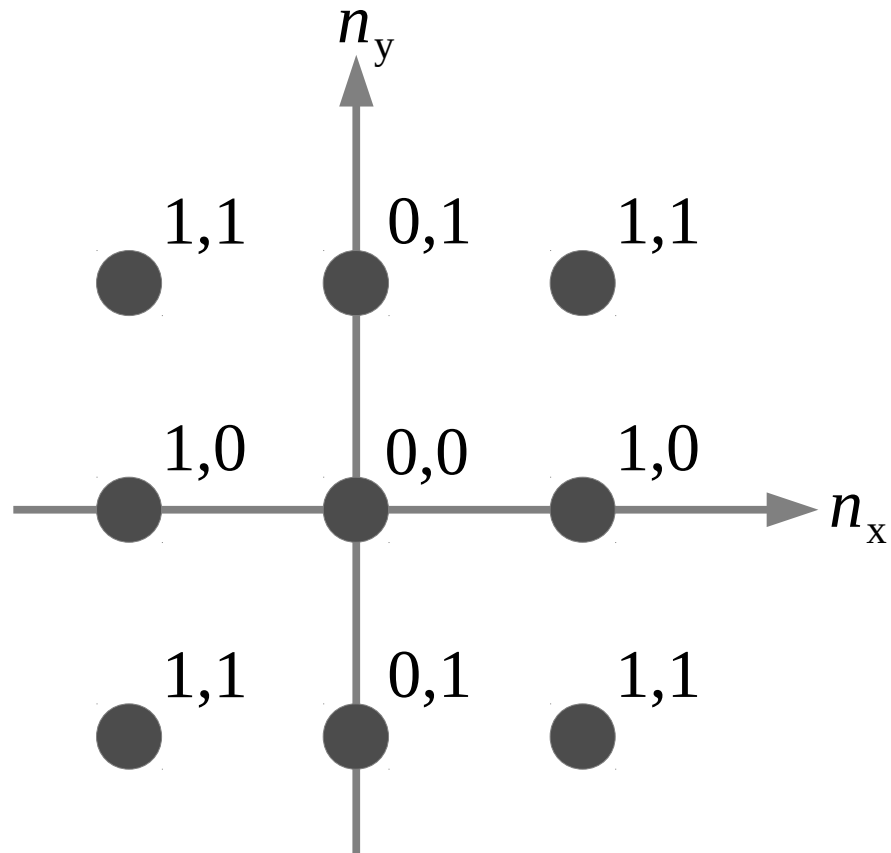


$$\Delta(\mathbf{r}) = \Delta_0 e^{i\mathbf{q}\cdot\mathbf{r}}$$

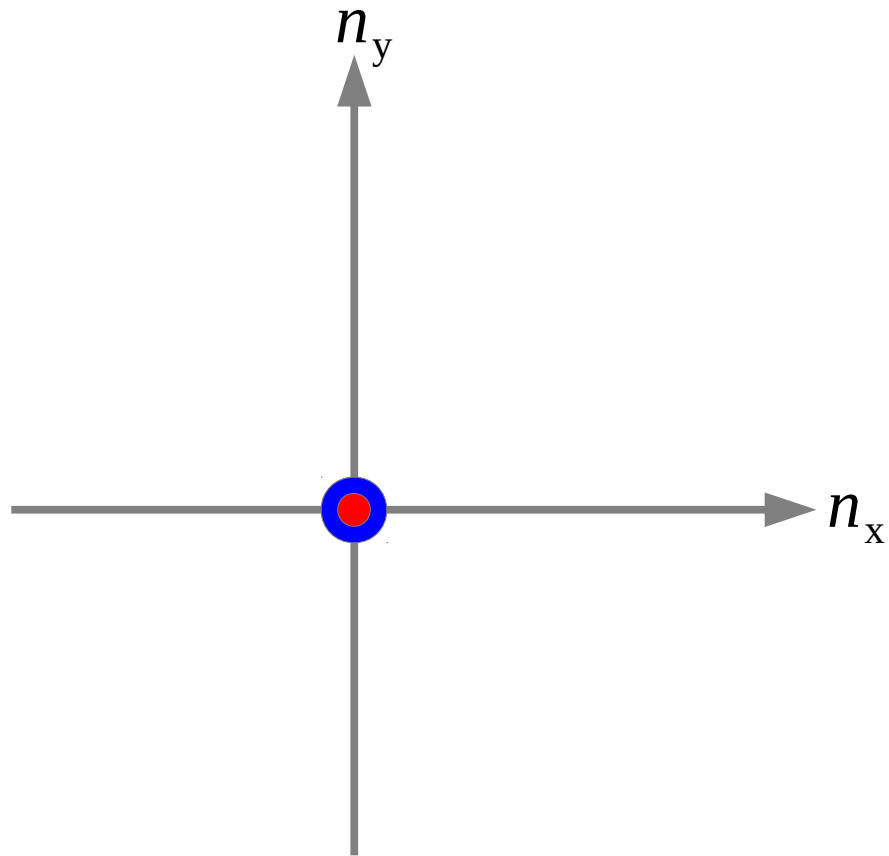
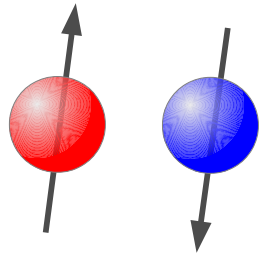
# Available states



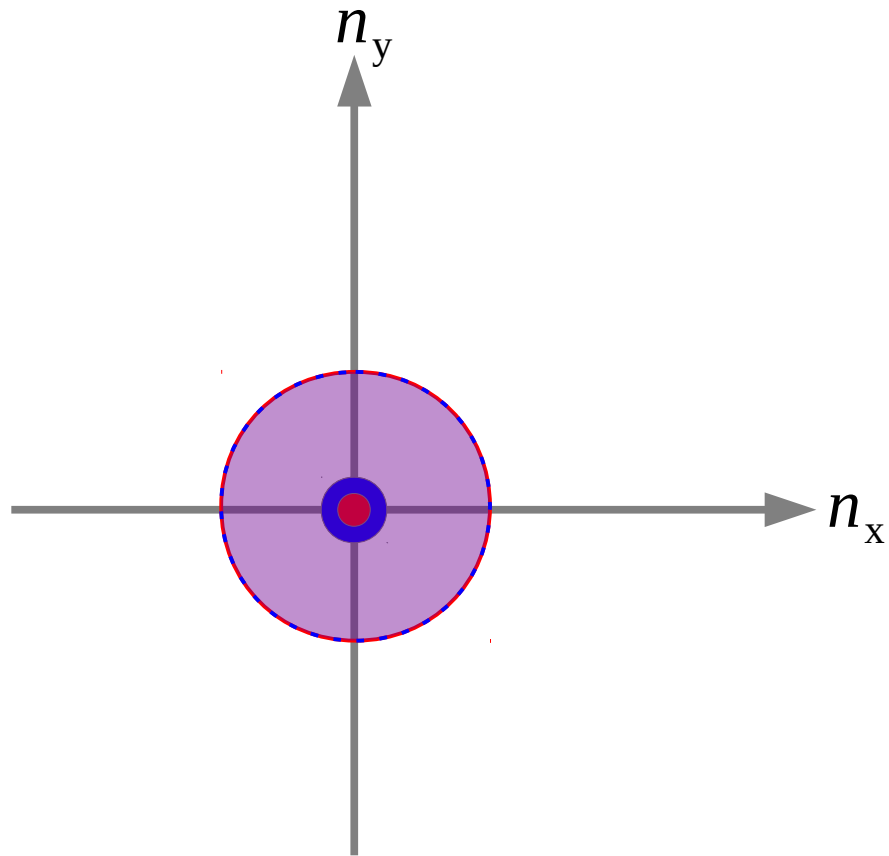
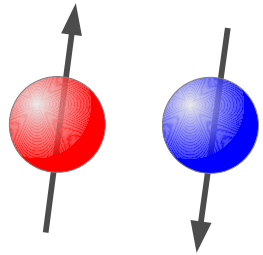
# Available states



# 2 atoms in a spherical trap

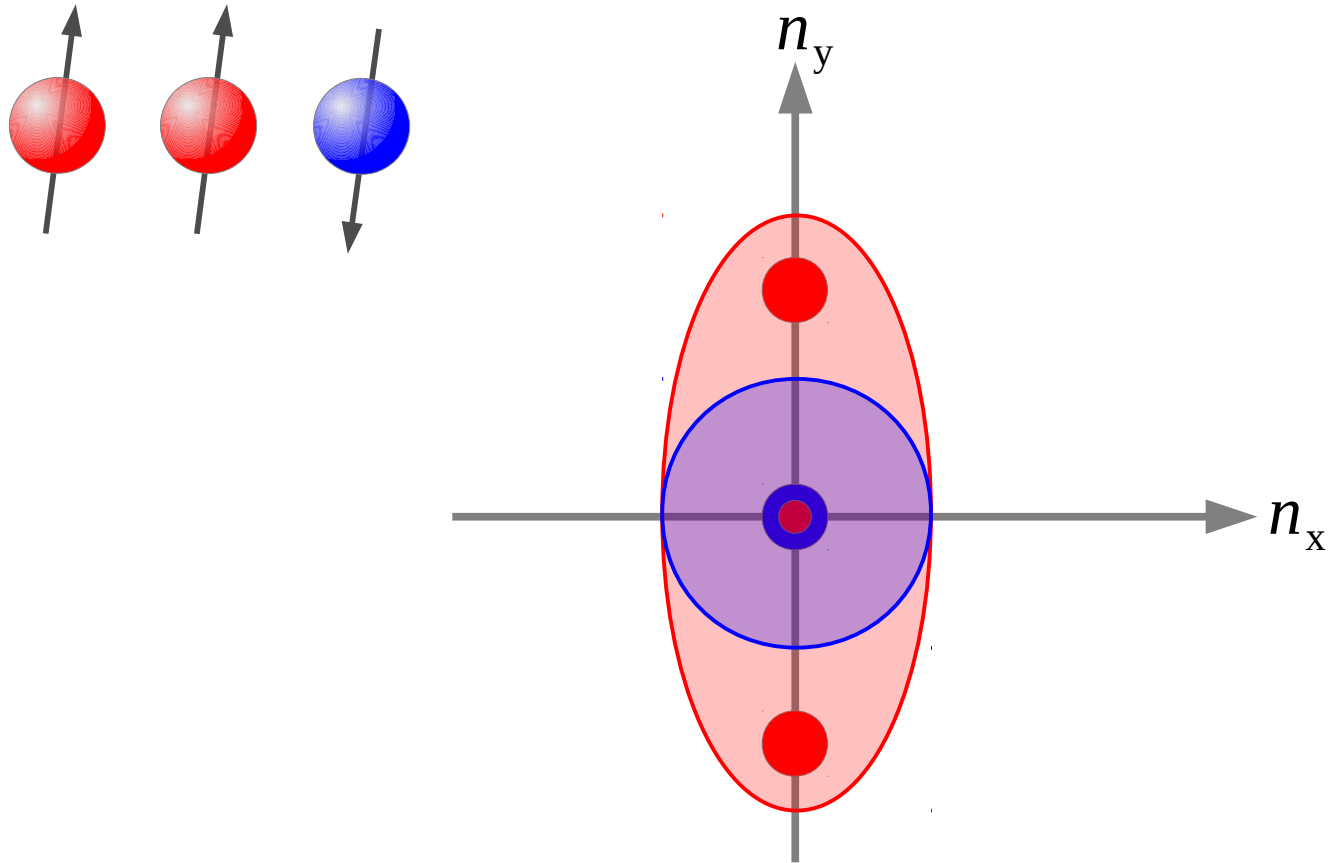


# 2 atoms in a spherical trap

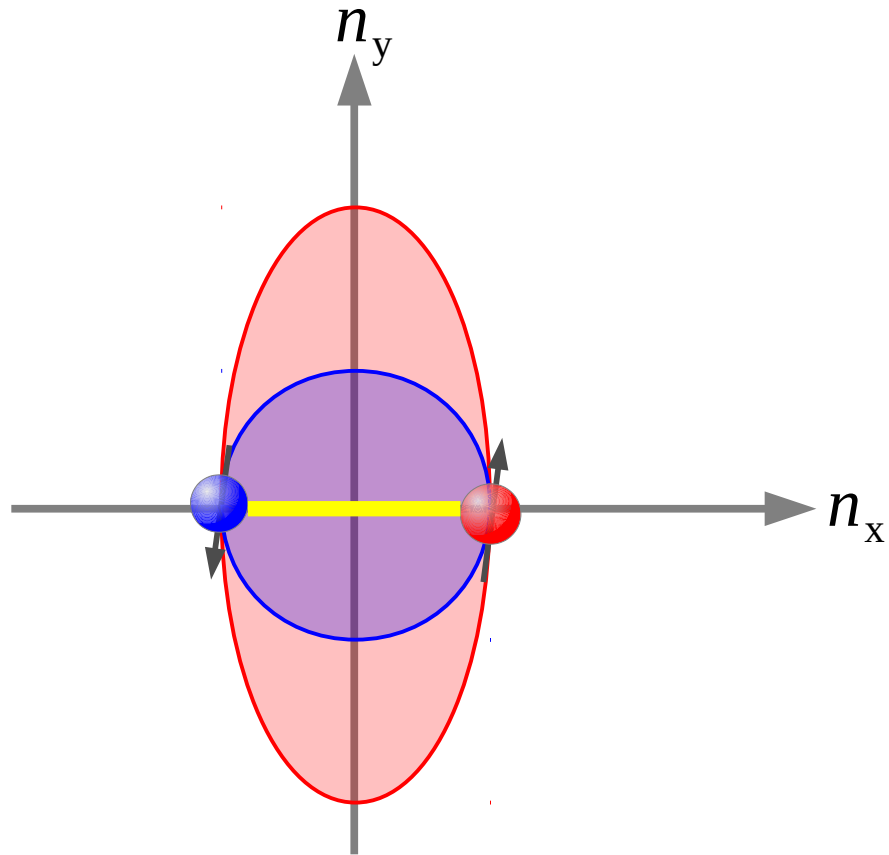




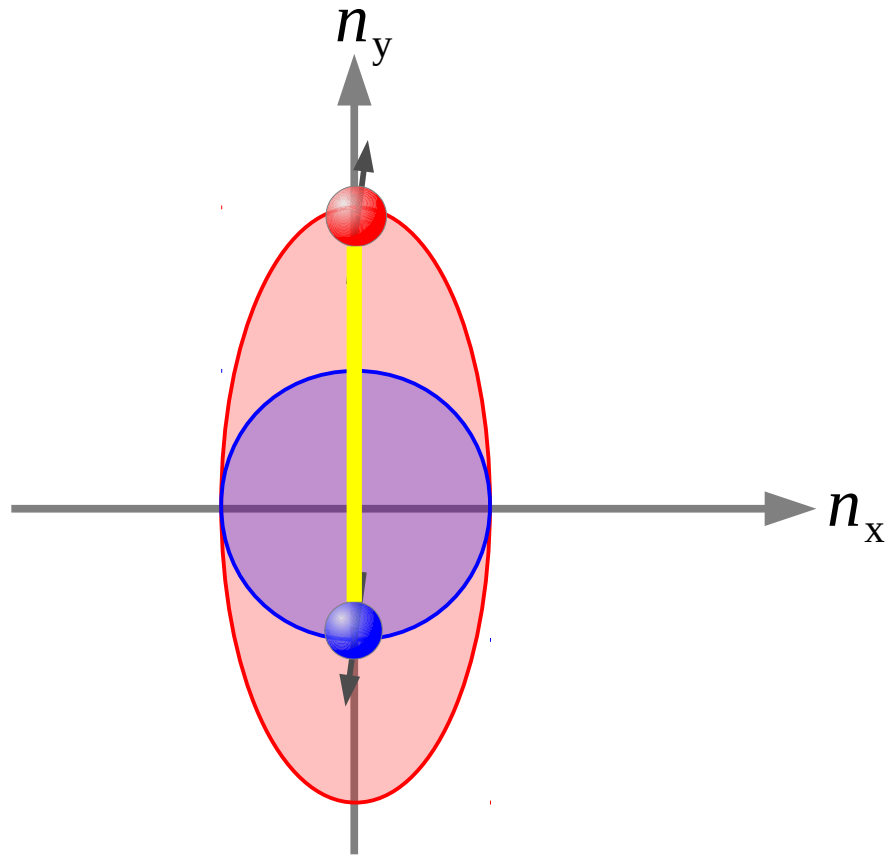
# 3 atoms in a spherical trap



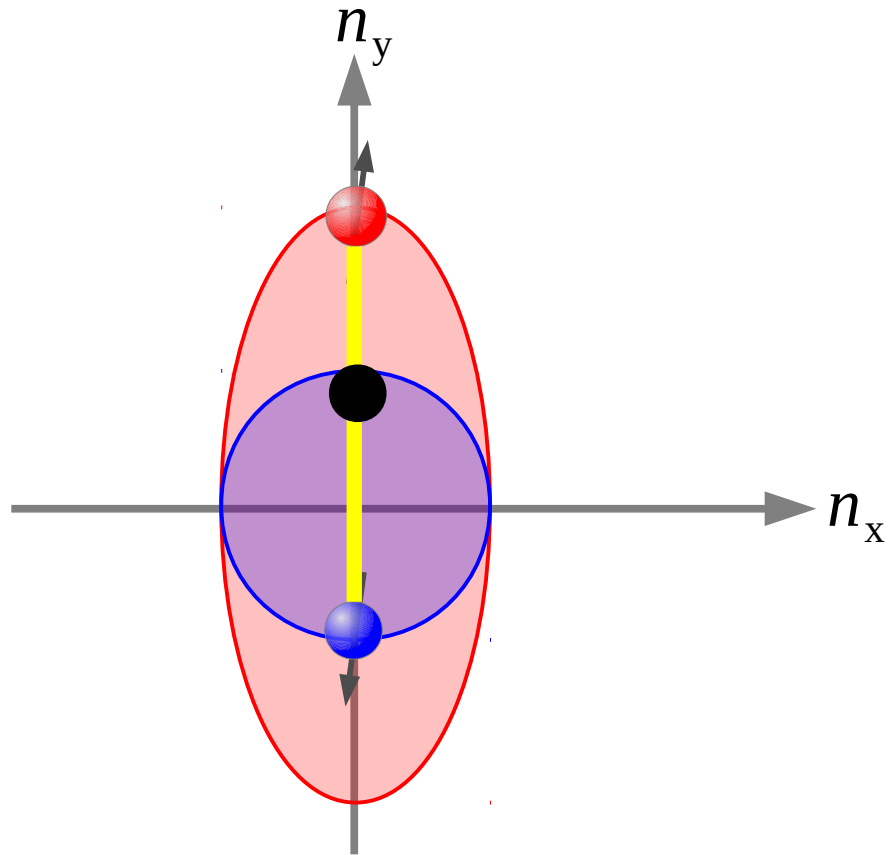
# 3 atoms in a spherical trap



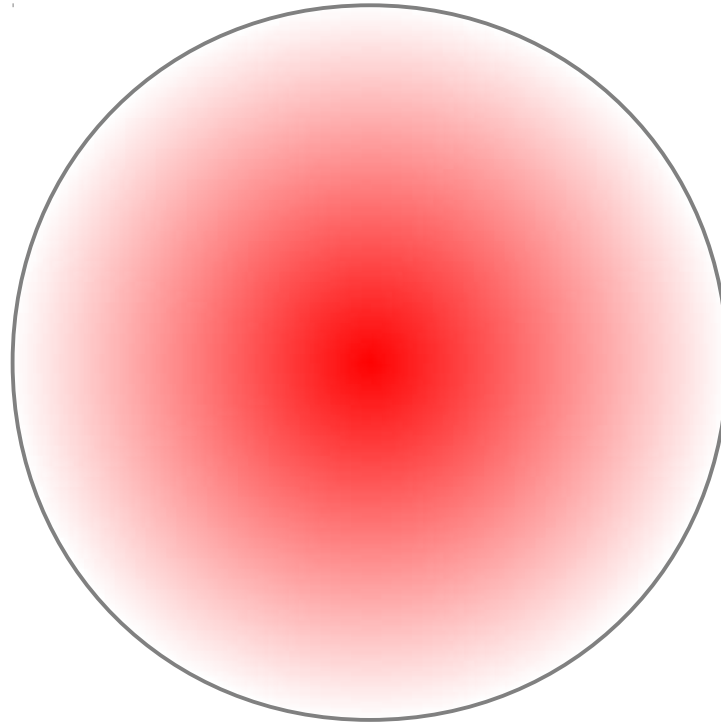
# 3 atoms in a spherical trap



# 3 atoms in a spherical trap

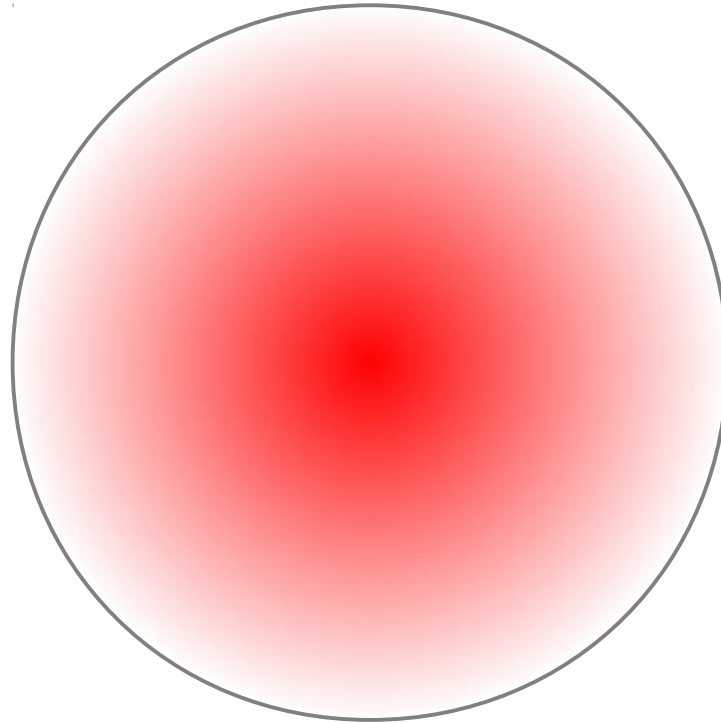


# 2 atoms in a spherical trap



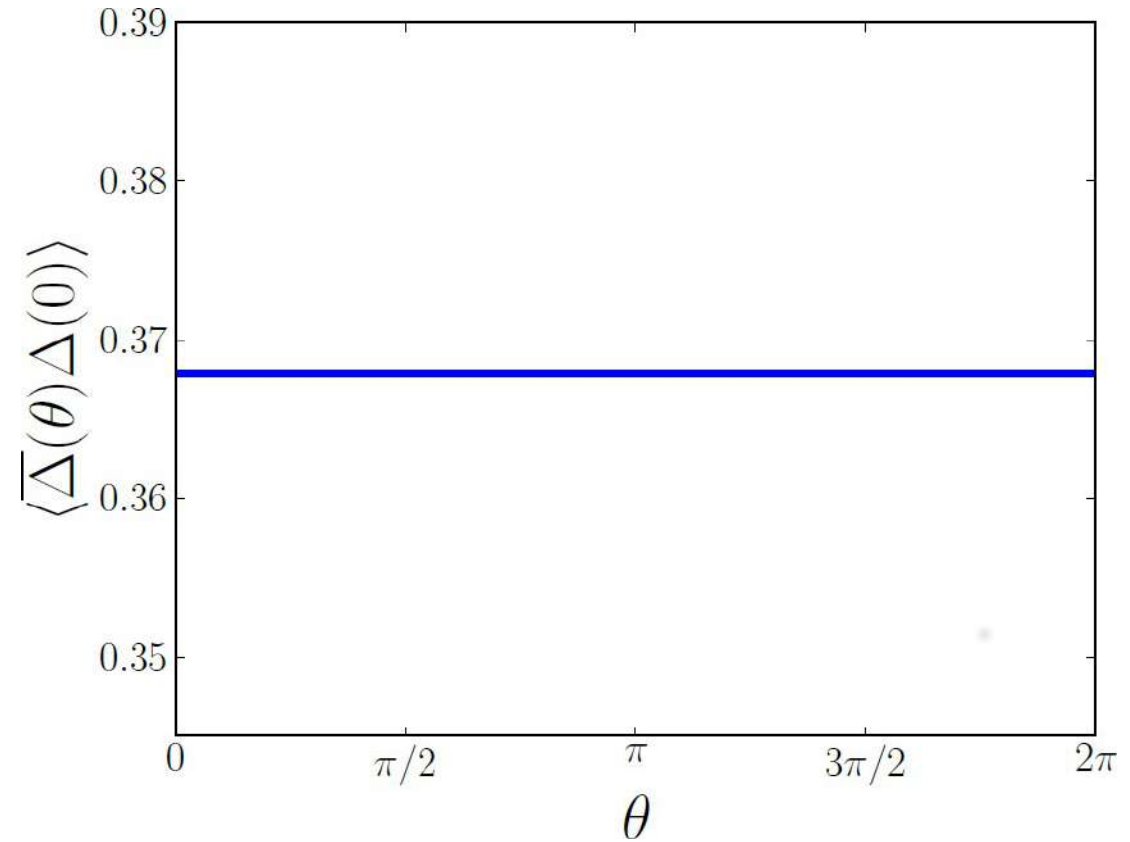
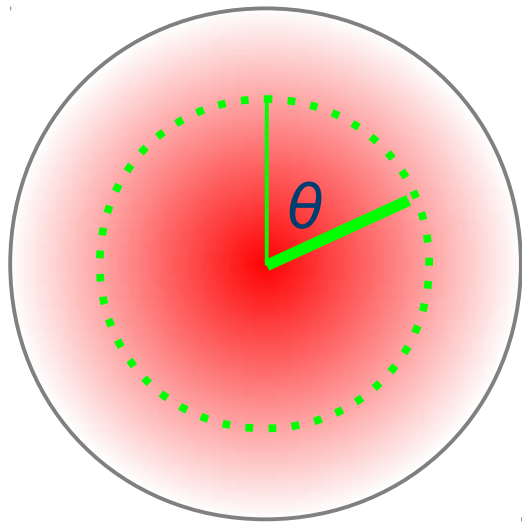
$$\Delta(\mathbf{r}) = \langle \psi | c_{\uparrow}(\mathbf{r}) c_{\downarrow}(\mathbf{r}) | \psi \rangle$$

## 2 atoms in a spherical trap

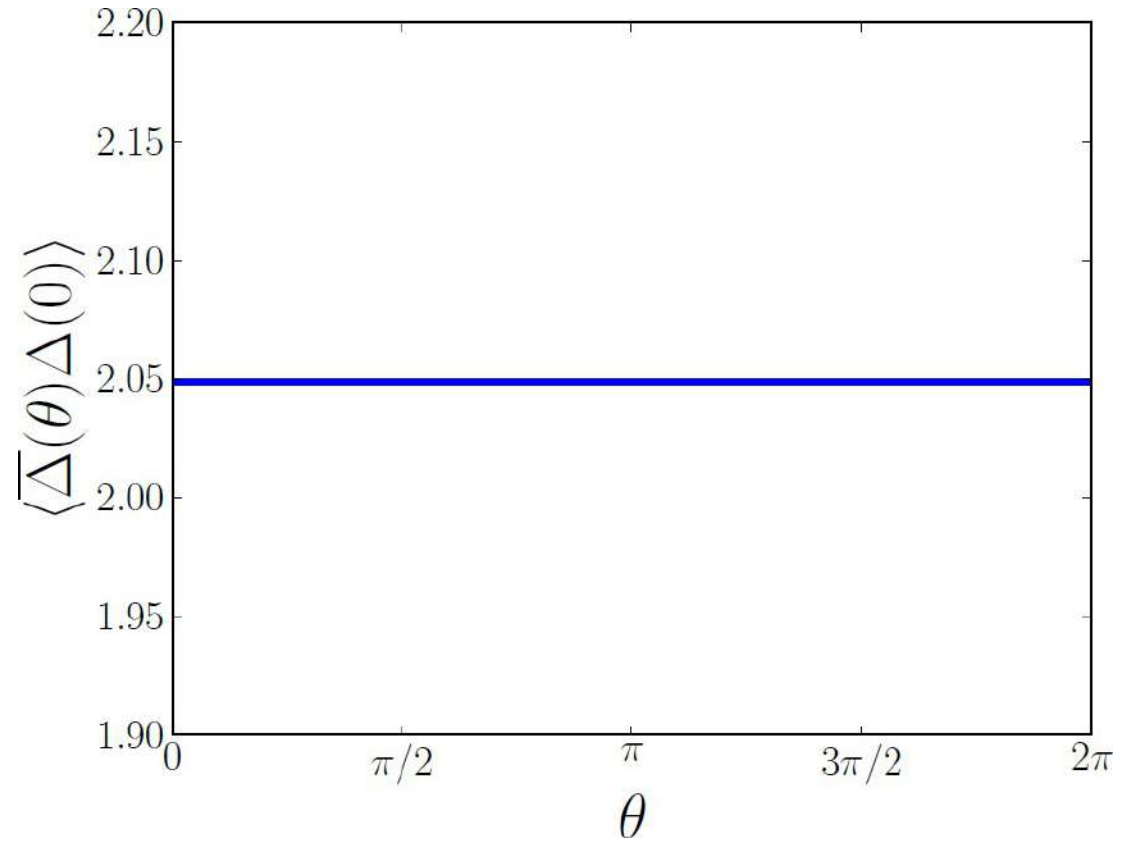
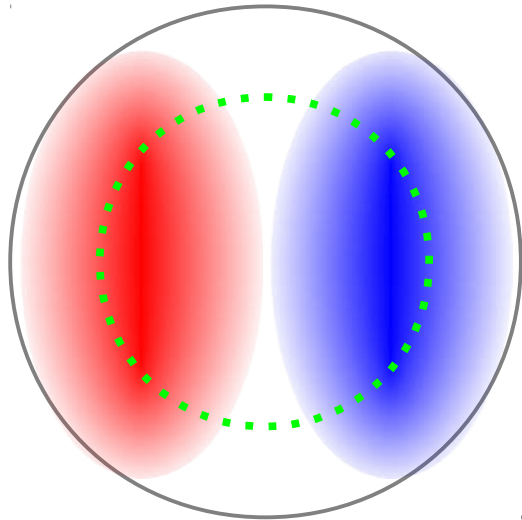


$$\bar{\Delta}(\mathbf{r})\Delta(\mathbf{0}) = \langle \psi | c_{\downarrow}^{\dagger}(\mathbf{r})c_{\uparrow}^{\dagger}(\mathbf{r})c_{\uparrow}(\mathbf{0})c_{\downarrow}(\mathbf{0}) | \psi \rangle$$

# 2 atoms in a spherical trap

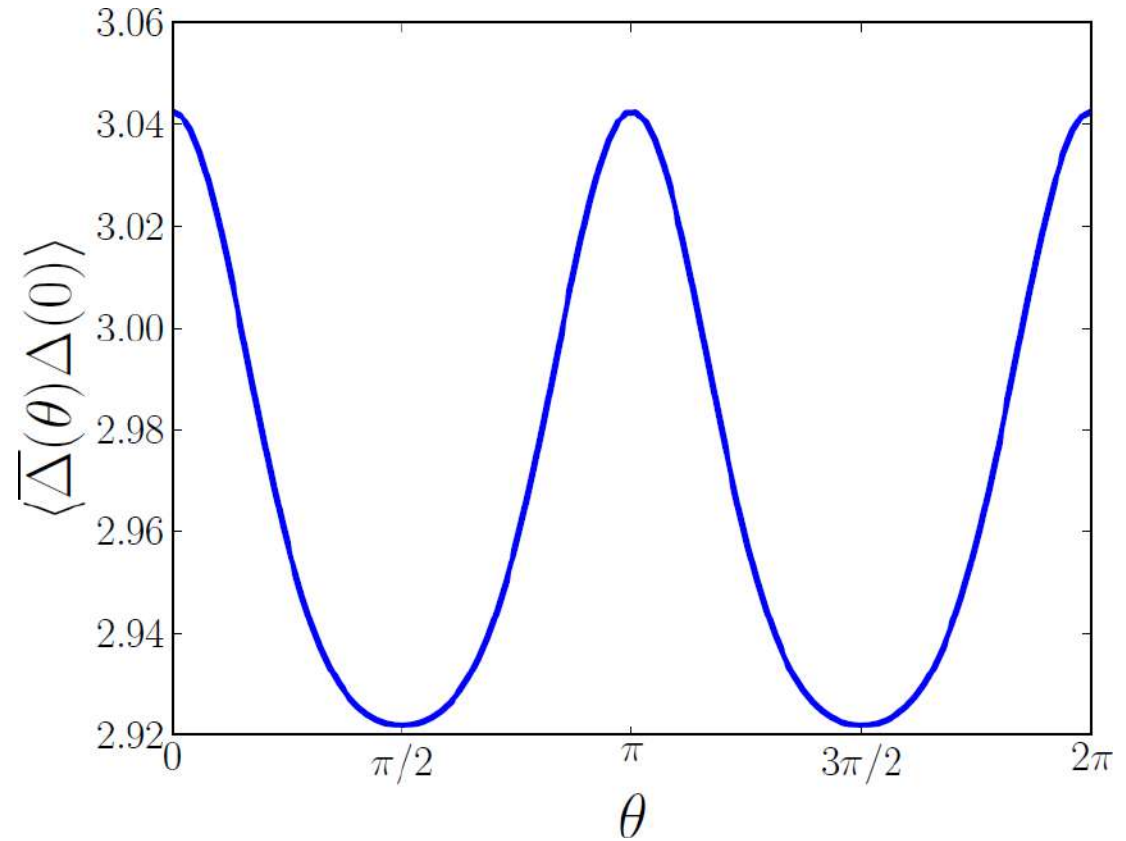
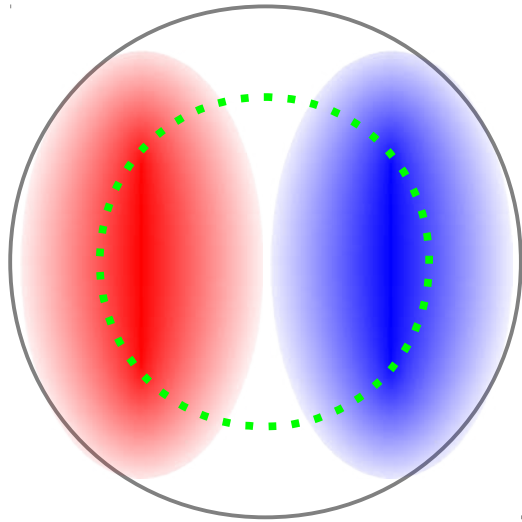


# 3 atoms in a spherical trap

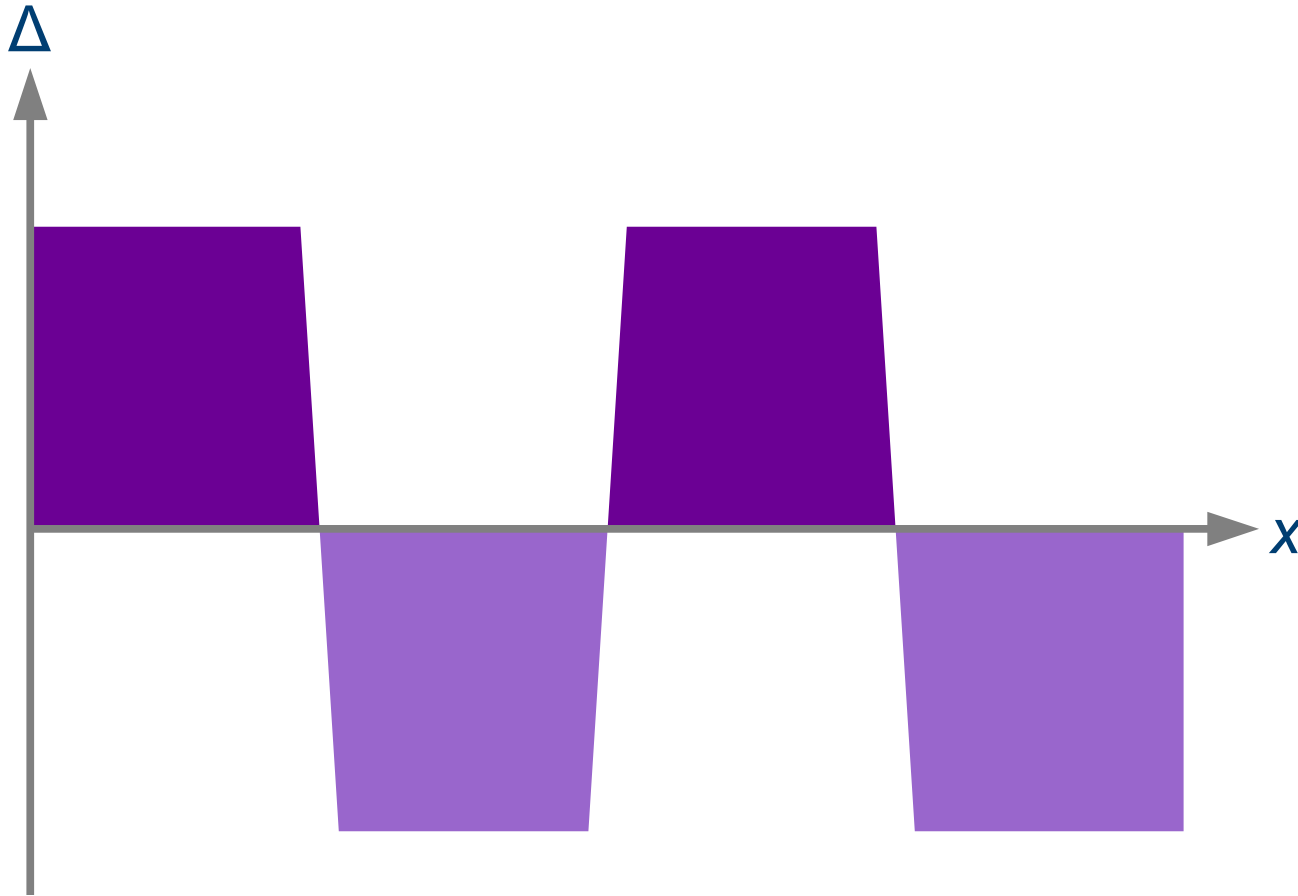




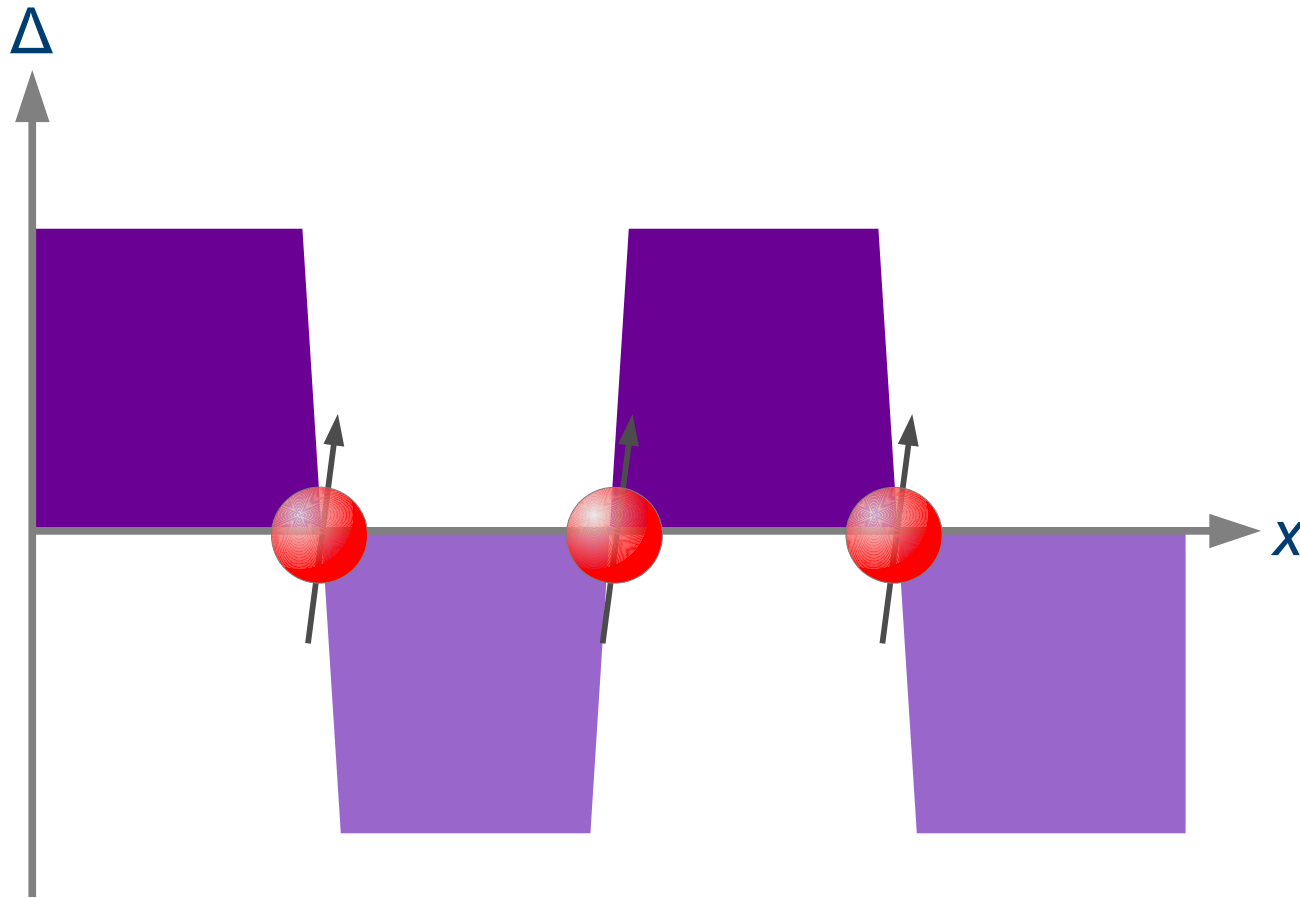
# 3 atoms in a spherical trap



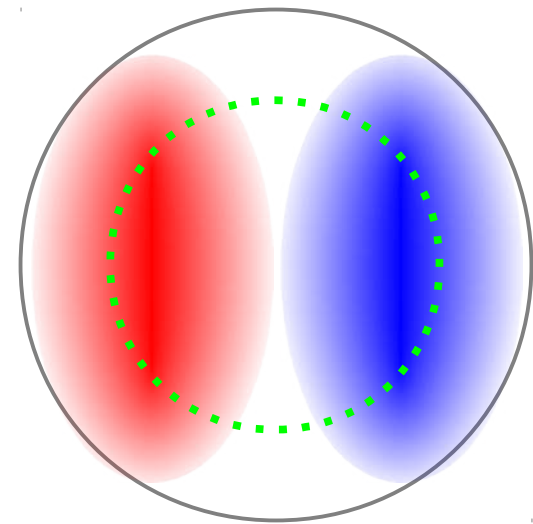
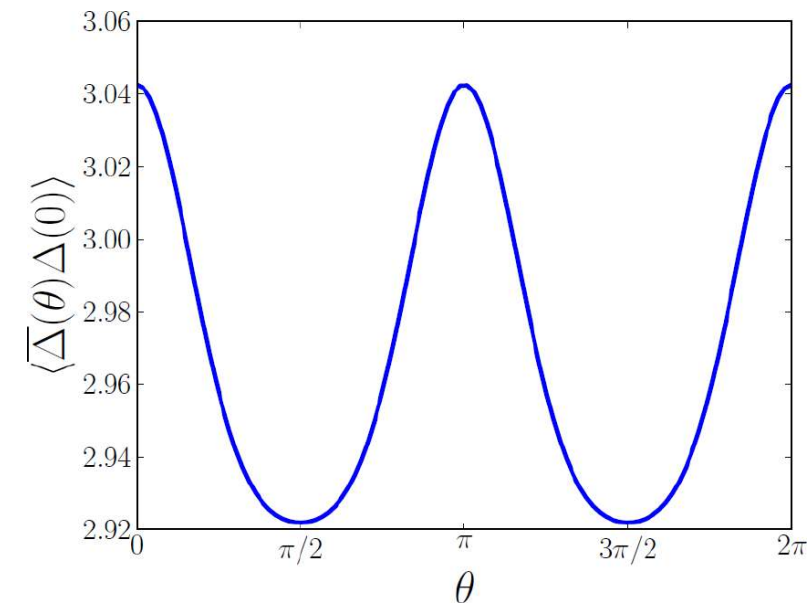
# Where does the extra majority spin reside?



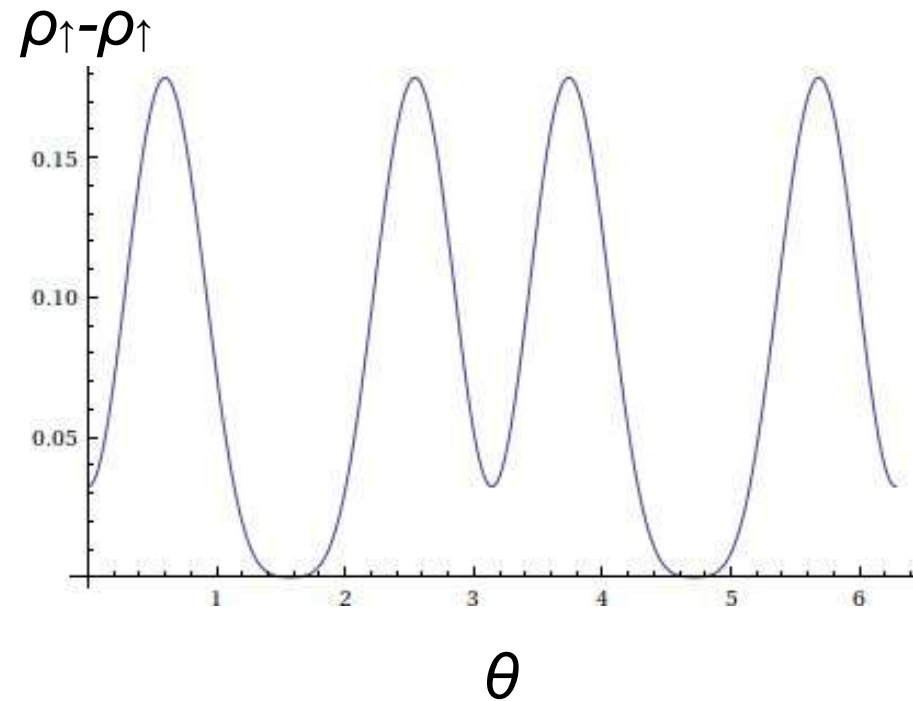
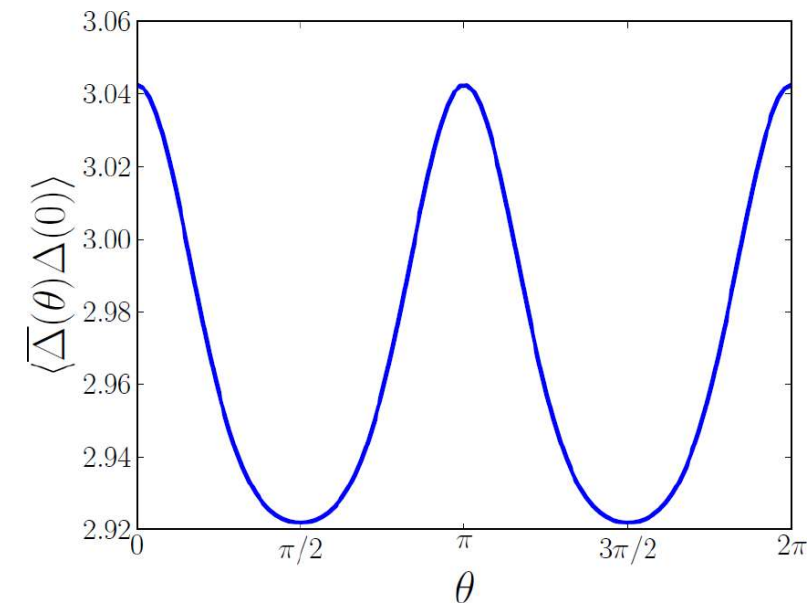
# Where does the extra majority spin reside?



# 3 atoms in a spherical trap

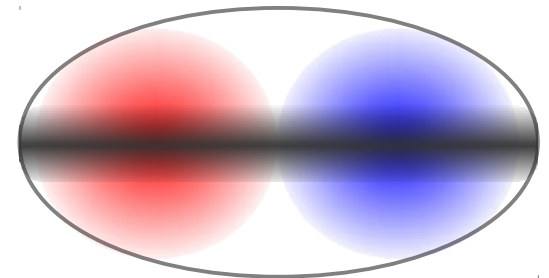
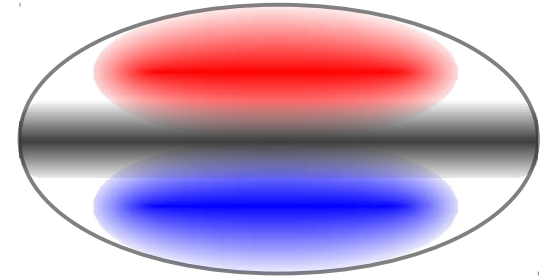
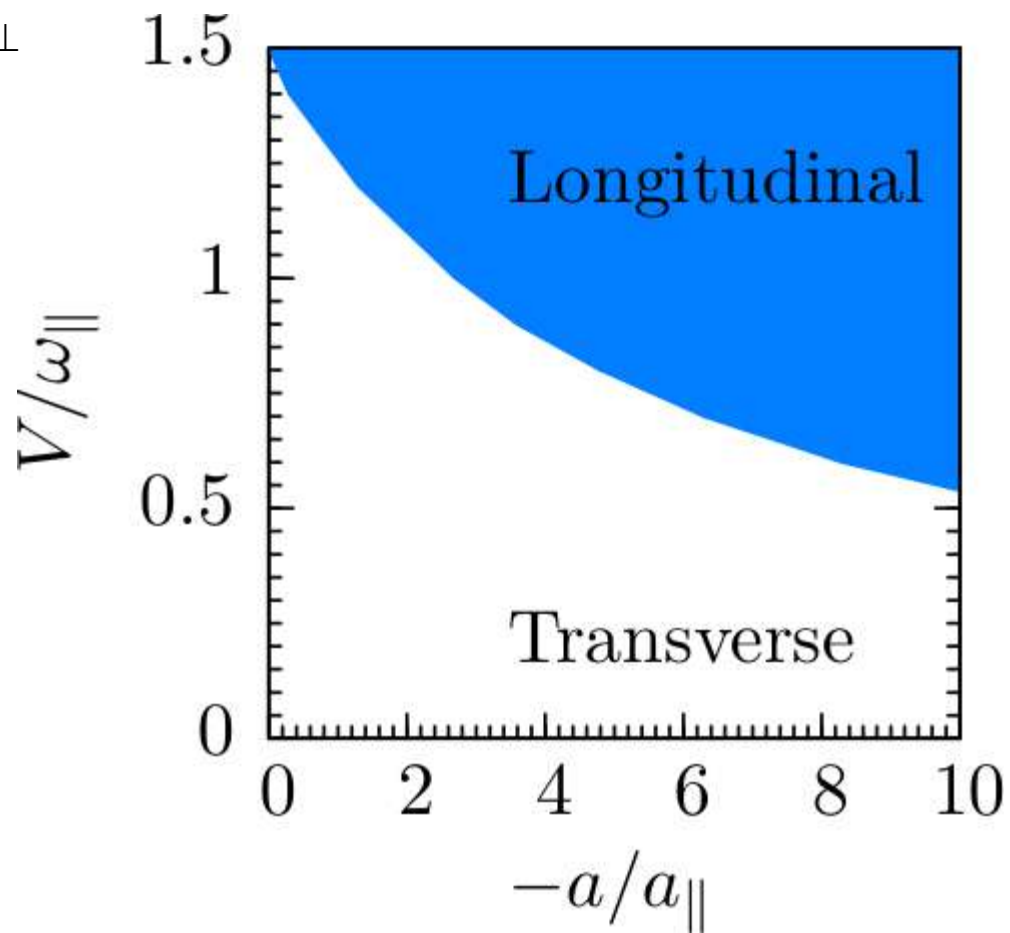


# 3 atoms in a spherical trap

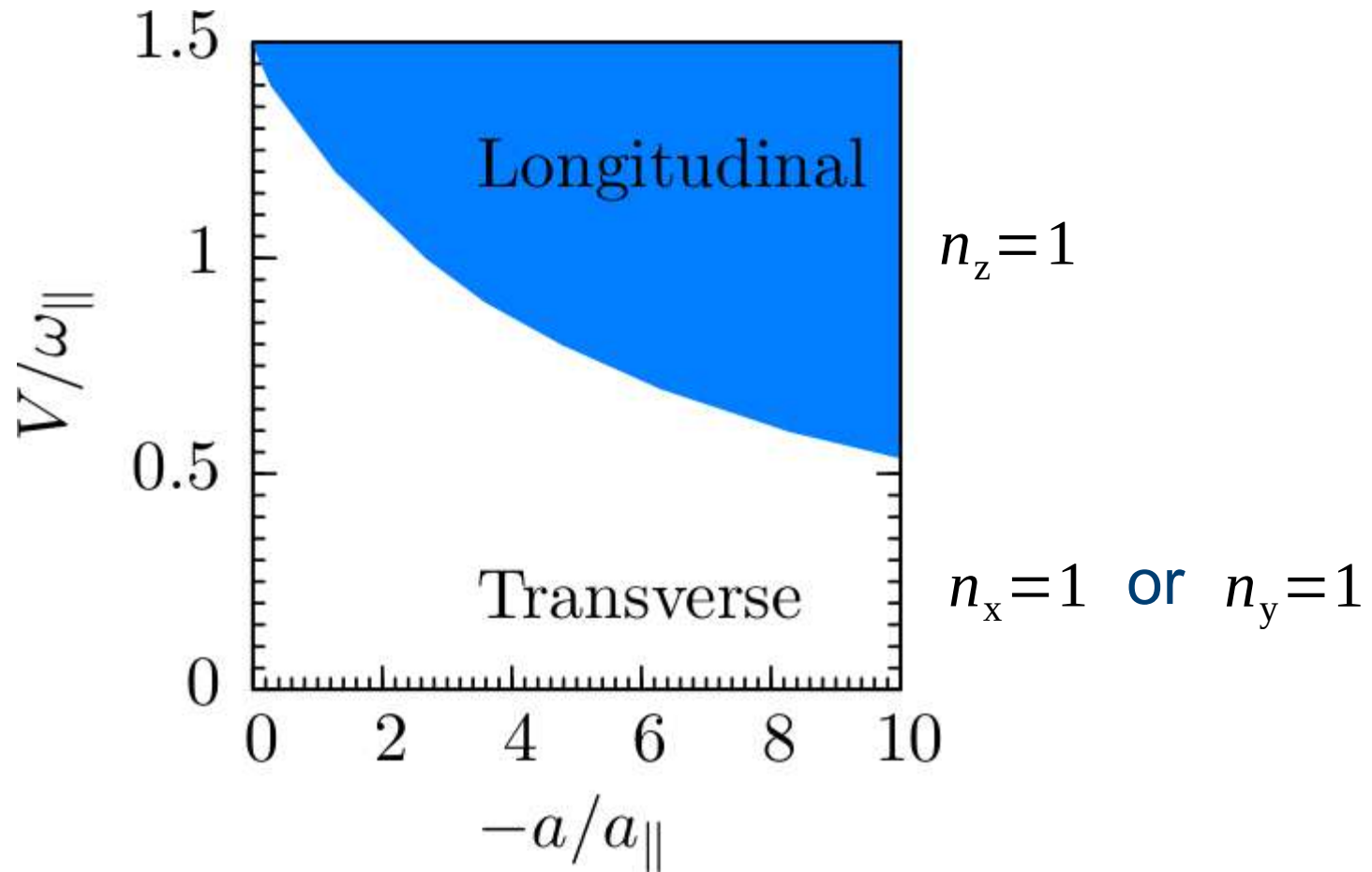


# Phase diagram

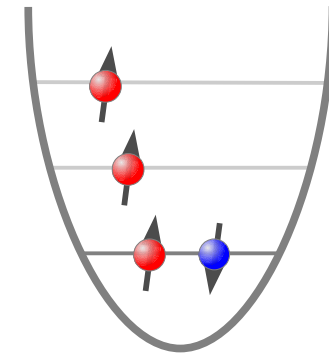
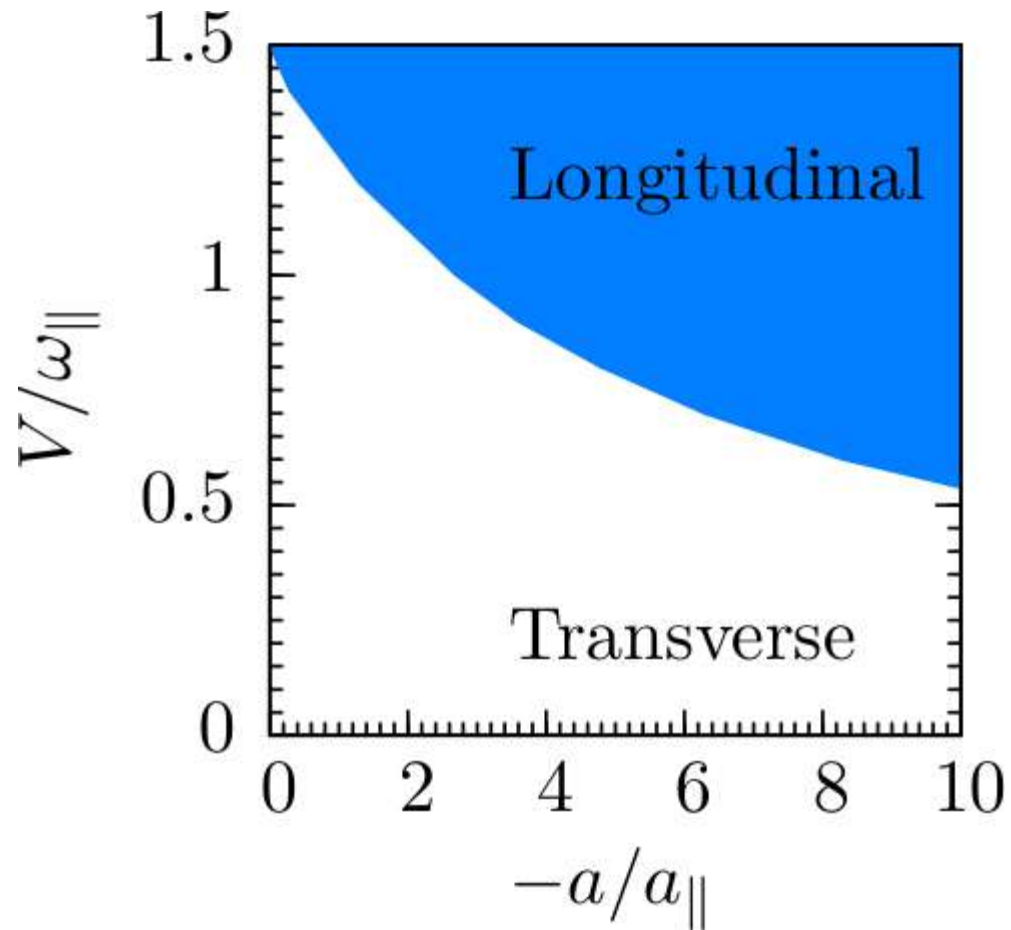
$$\omega_{\parallel} = 2\omega_{\perp}$$



# Phase diagram

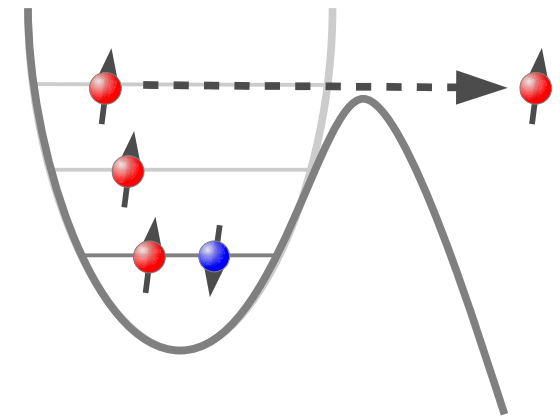
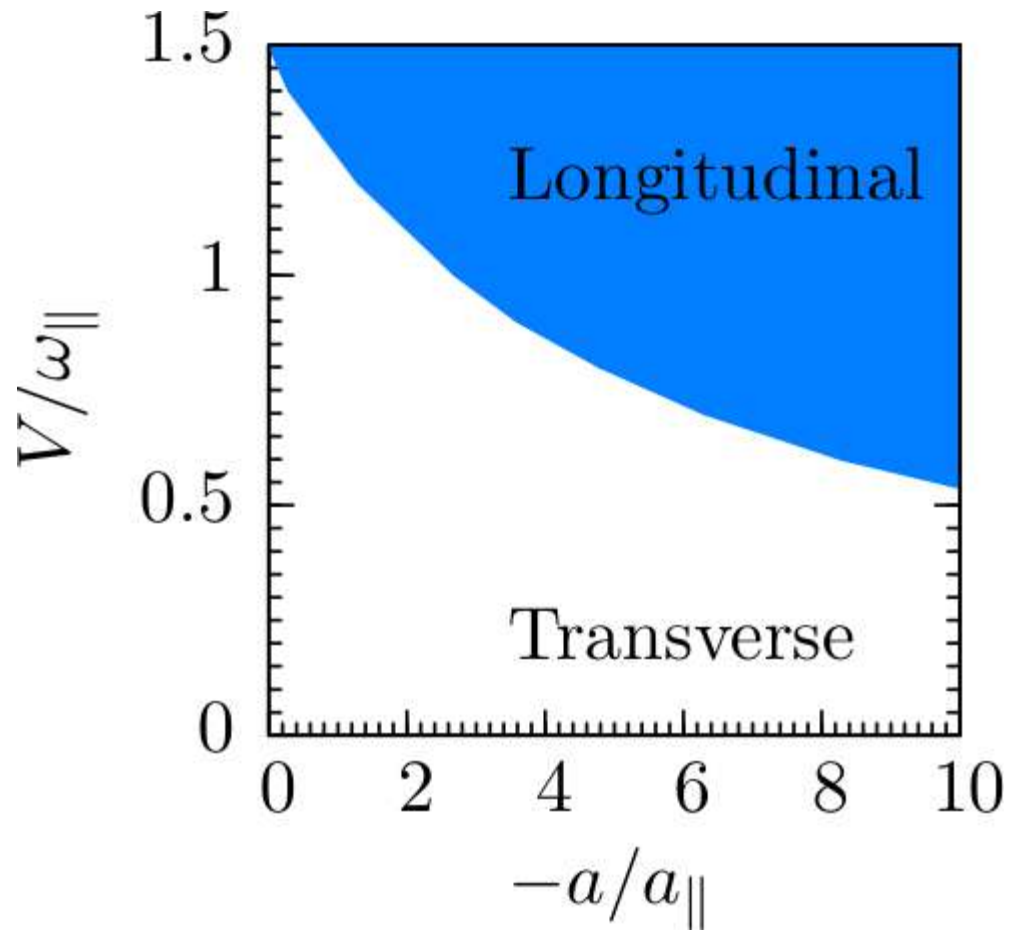


# Phase diagram

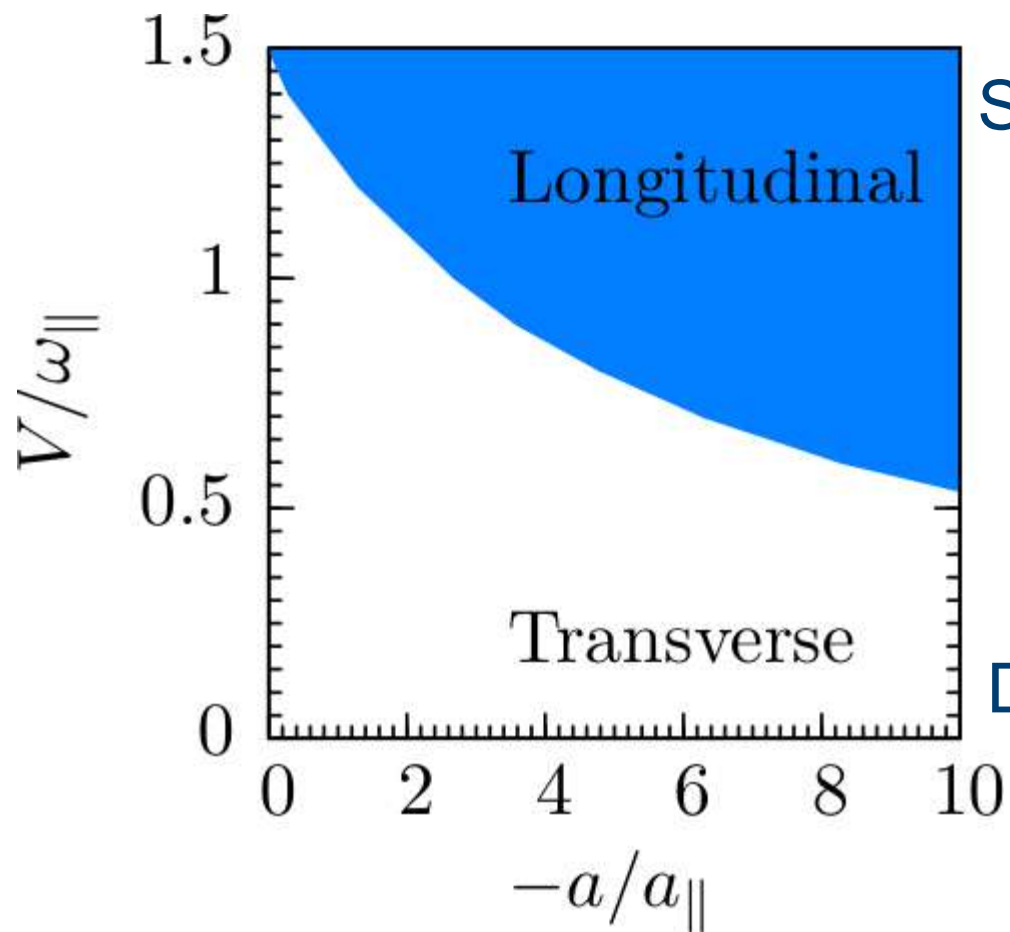




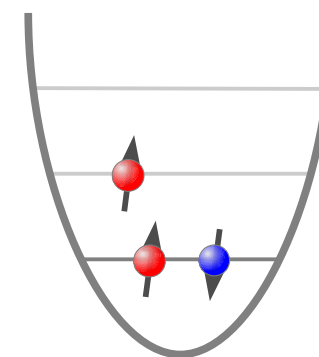
# Phase diagram



# Phase diagram



Singly degenerate



Doubly degenerate

# Summary

Few trapped fermions offers chance to observe spatially modulated pairing

Trap ellipticity and central barrier are experimental probes of the pairing state

# Appendix: System Geometry

$$g(n) = (n + 1)(n + 2)/2$$

$$N \sim \int_0^{n_{max}} dn g(n) \simeq n_{max}^3/6$$

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Hamiltonian for outermost electron

$$\langle \hat{H}_{n_{max}} \rangle \simeq \frac{1}{2} m \omega^2 \langle \hat{r}^2 \rangle$$

$$\langle \hat{H}_{n_{max}} \rangle = \hbar \omega \left( n_{max} + \frac{3}{2} \right)$$

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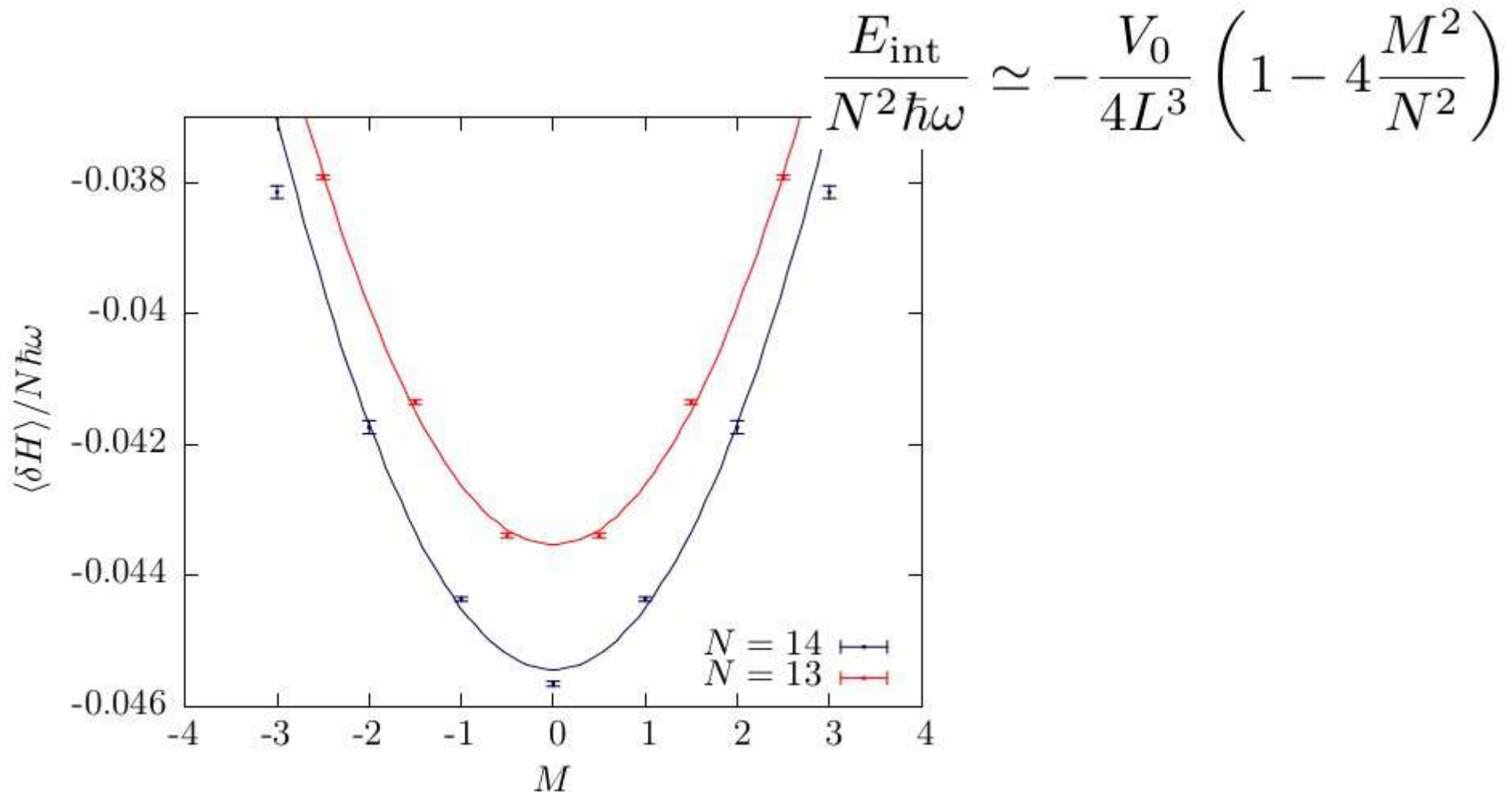
$$L^3 = \frac{4}{3} \pi \langle \hat{r}^3 \rangle \sim N^{1/2}$$

$$\frac{E_{\text{int}}}{N^2} = -\frac{V_0}{4L^3}$$

$$\frac{E_{\text{int}}}{N^2} \sim -V_0 N^{-1/2}$$

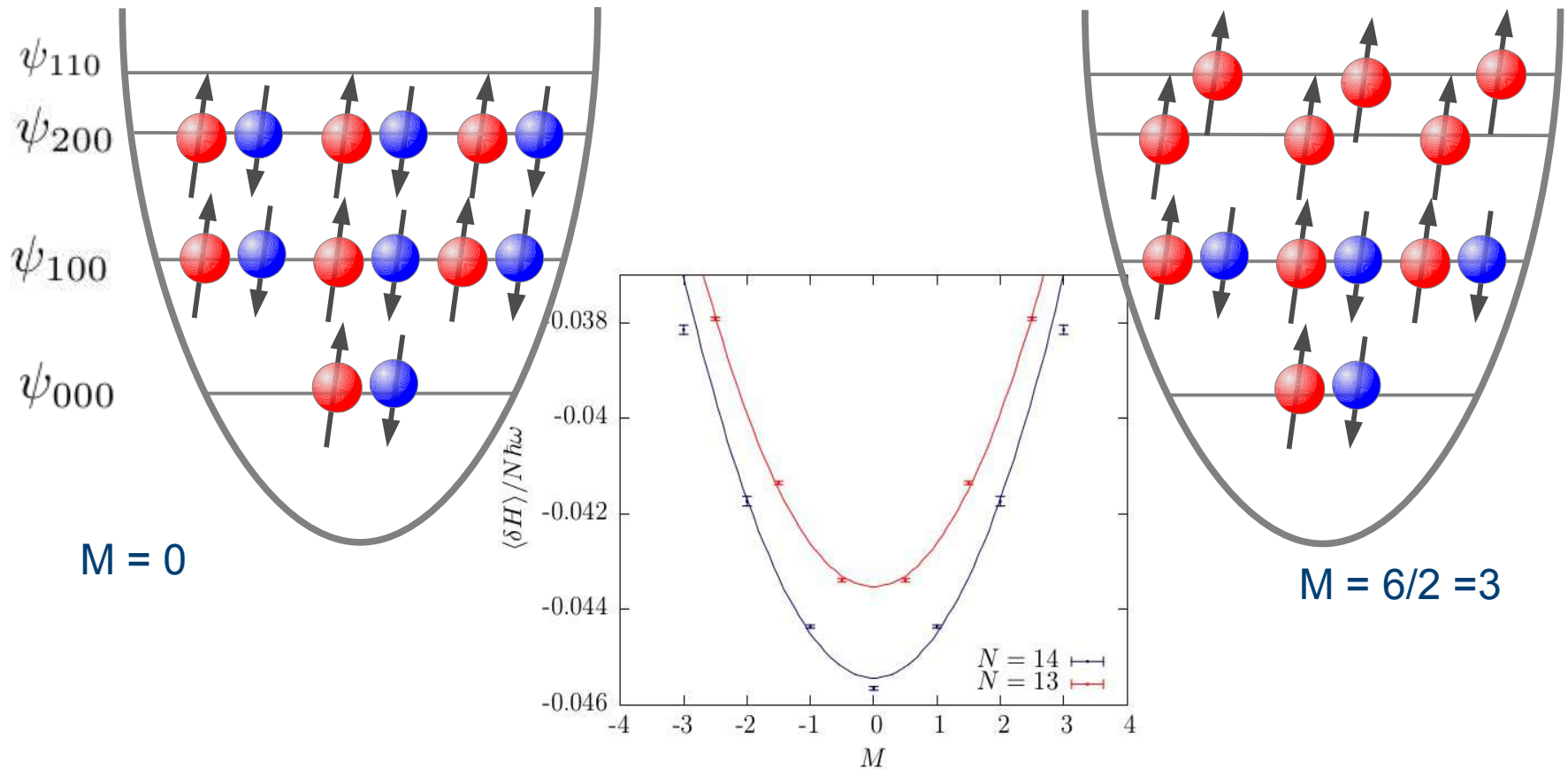


# Appendix: Magnetised Fermionic Gases



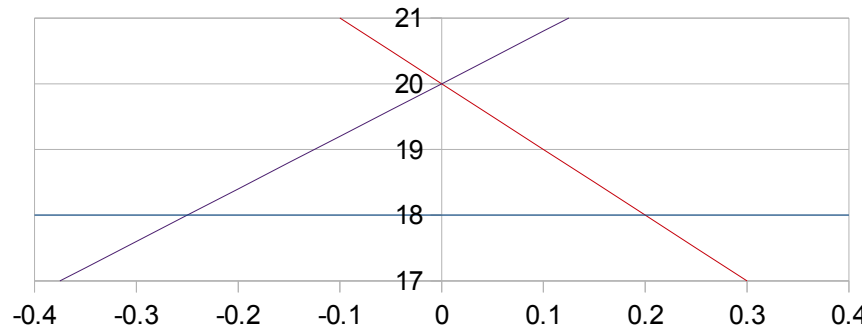
# Appendix: Magnetised fermionic gases

- $N = 14$



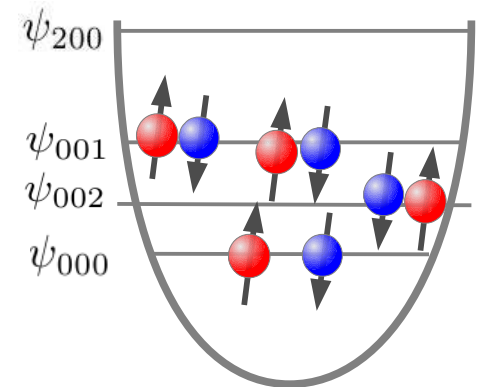
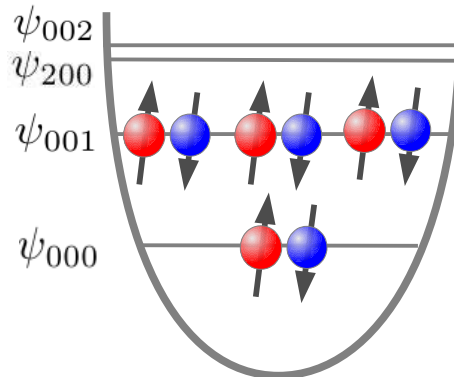
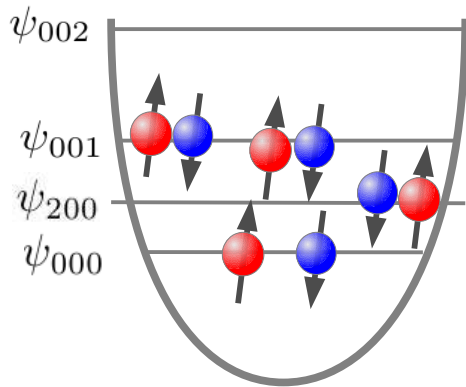
# Appendix: Asymmetric Trap

$$\hat{H}^{(0)} = \frac{-\hbar^2}{2m} \nabla^2 + \frac{1}{2} m (\omega_{\perp}^2 x^2 + \omega_{\perp}^2 y^2 + \omega_{\parallel}^2 z^2)$$

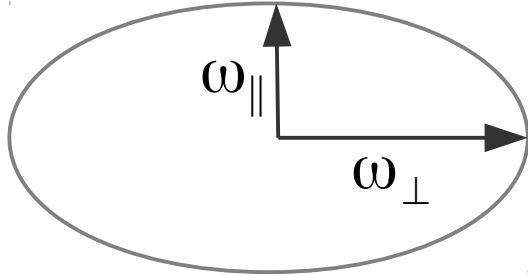


$$\omega_{\parallel}(s) = \omega_0(1 - 2s)$$

$$\omega_{\perp}(s) = \omega_0(1 + s)$$

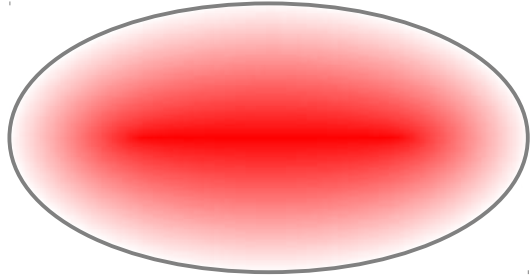


# Trapping potential



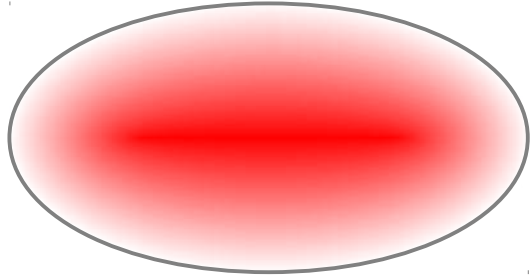
$$V = \frac{1}{2} [\omega_{\perp}^2 (x^2 + y^2) + \omega_{\parallel}^2 z^2]$$

# One trapped atom



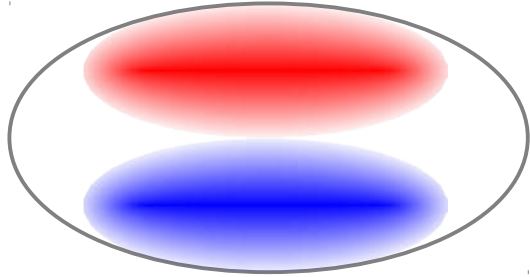
$$E = \frac{1}{2}\omega_{\parallel} + \omega_{\perp}$$

# Two trapped atoms

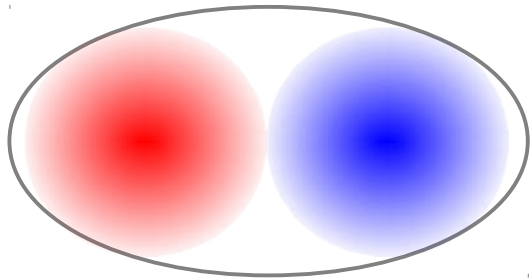


$$E = \omega_{\parallel} + 2\omega_{\perp}$$

# Three trapped atoms

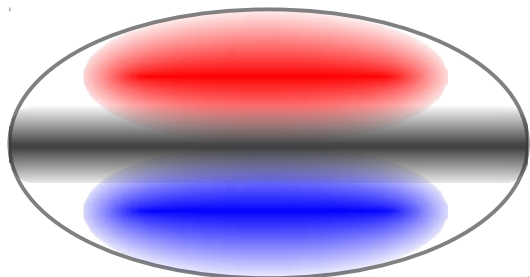


$$E = \frac{5}{2}\omega_{\parallel} + 3\omega_{\perp}$$

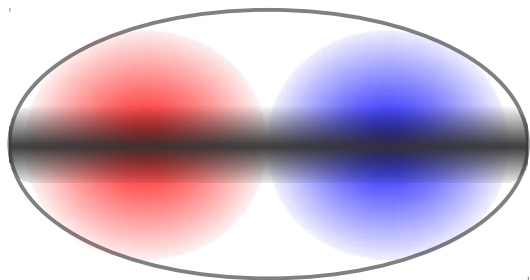


$$E = \frac{3}{2}\omega_{\parallel} + 4\omega_{\perp}$$

# Three trapped atoms



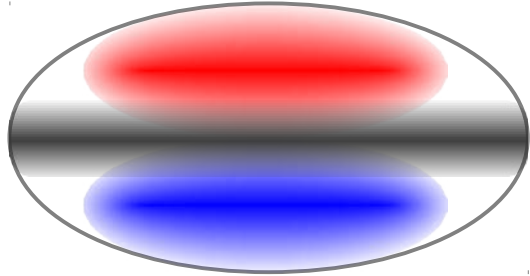
$$E = \frac{5}{2}\omega_{\parallel} + 3\omega_{\perp} + \sqrt{\frac{\omega_{\parallel}}{\omega_{\parallel} + \omega_B}} V_B \left( 2 + \frac{\omega_{\parallel}}{\omega_{\parallel} + \omega_B} \right)$$



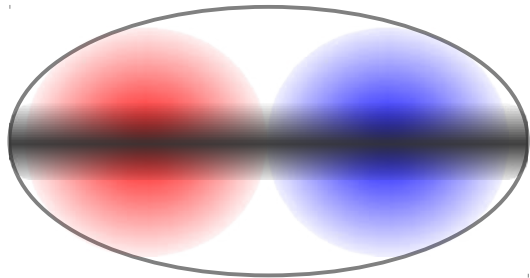
$$E = \frac{3}{2}\omega_{\parallel} + 4\omega_{\perp} + 3\sqrt{\frac{\omega_{\parallel}}{\omega_{\parallel} + \omega_B}} V_B$$



# Three trapped atoms



$$E = \frac{5}{2}\omega_{\parallel} + 3\omega_{\perp} + \sqrt{\frac{\omega_{\parallel}}{\omega_{\parallel} + \omega_B}} V_B \left( 2 + \frac{\omega_{\parallel}}{\omega_{\parallel} + \omega_B} \right) + \frac{a}{a_{\parallel}} \omega_{\perp} \sqrt{\frac{2}{\pi}} \left( \frac{3}{2} - \frac{4\sqrt{2}}{\pi} \frac{V_B}{\omega_{\parallel} + \omega_B} \right)$$



$$E = \frac{3}{2}\omega_{\parallel} + 4\omega_{\perp} + 3\sqrt{\frac{\omega_{\parallel}}{\omega_{\parallel} + \omega_B}} V_B + \frac{a}{a_{\parallel}} \omega_{\perp} \sqrt{\frac{2}{\pi}} \left( \frac{3}{2} - \frac{6\sqrt{2}}{\pi} \frac{V_B}{\omega_{\parallel} + \omega_B} \right)$$