Fine Structure of the Phonon in One Dimension from Quantum Hydrodynamics

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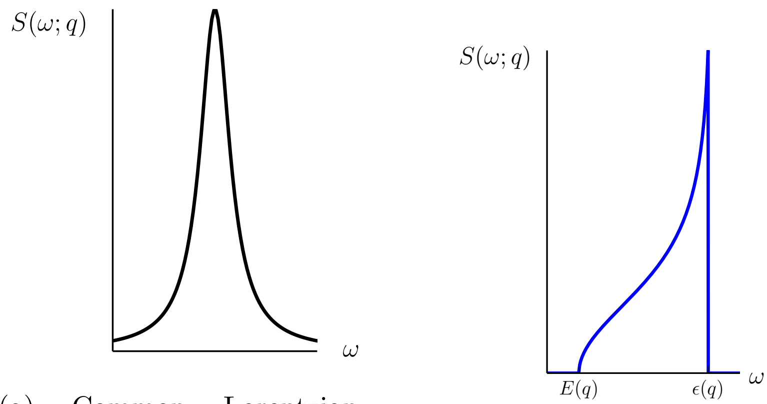
1 The Problem: Phonon Lineshape in 1D

All known matter vibrates when subject to a driving force, be it a wine glass near an opera singer, or an exotic new material in the lab. Near zero temperature, the amplitude of this response as a function of the driving frequency provides a window on the quantum mechanical spectrum of a material's vibrations.

If the dynamics of the material could be described by noninteracting phonons with a wavenumber dependent energy $\epsilon(q)$, then the material would only ever vibrate if $\hbar\omega$ were exactly resonant with $\epsilon(q)$. Of course, this singular response is an artefact of an oversimplified model. In reality, we see a nonzero response over a nonzero range of frequencies. This broadening of the infinitely sharp peak into its "fine structure" is a result of the finite lifetime of phonons, because anharmonicity couples different phonon modes. It is the self–respecting theorist's job to calculate this lineshape.



(a) Common Lorentzian lineshape at zero temperature for dimensions $d > 1$.

(b) 1D phonon lineshape at zero temperature (interactions included), after [1].

Figure 1: Gallery of phonon lineshapes.

Phonon interactions in 1D are singular, because two phonons with the same speed never move apart, and the textbook perturbative scattering approaches that would lead to the Lorentzian lineshape of Fig. 1a are inapplicable. Our result is that including the phonon dispersion leads to a nonlinear quantum field theory that can be analysed in the limit of strong dispersion [1]. Specifically, the phonon lineshape acquires the power law singularities seen in Fig. 1b.

I should point out that the seemingly oversimple description of 1D fluids in terms of noninteracting, nondispersing phonons has enjoyed many successes over the past 50 years.

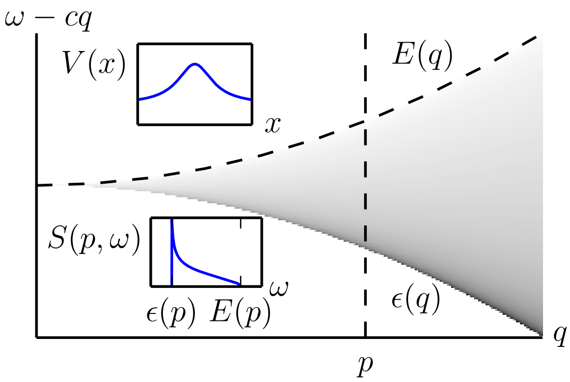


Figure 2: Dispersion relations of the phonon $\epsilon(q)$ and soliton $E(q)$, as seen in the frame moving with the linear speed of sound c. The amplitude of the density response $S(p,\omega)$ is indicated in greyscale. Upper Inset: Density profile of a soliton. Lower Inset: amplitude of the density response across the cut indicated at momentum $\hbar p$.

This is because of a remarkable map from strongly interacting microscopic particles (say electrons in a very narrow wire or a highly anisotropic crystal) onto "weakly" interacting phonons. For a long time it was thought that interactions between the *phonons* were irrelevant for all low frequency observables. For the phonon lineshape, this is not true, which left us the question of how to calculate the lineshape from a nonlinear theory of phonons.

2 Solution: Phonon Dispersion and Interactions

To resolve the delta function into the true lineshape it is necessary to include interactions between phonons. As well as interactions, we must include the fact that phonons will disperse with wavelength. This dispersion causes a localized wavepacket to fall apart, since its Fourier constituents all move with their own velocity. The remarkable fact, discovered at the turn of the last century, is that the interactions between the phonons oppose this effect and allow for solitary waves, or "solitons", to retain their form. We can calculate the energy E of the soliton as a function of its momentum $\hbar q$ to arrive at its dispersion relation, as indicated in Fig. 2.

To find the lineshape we need to study the *quantum mechanical* propagation of a soliton of momentum $\hbar q$ for a time t and take a Fourier transform over t. Feynman tells us to write the amplitude for a soliton to propagate for a time t with wavenumber q as a sum of $e^{iS[\phi]/\hbar}$ over all "paths" ϕ, where the phase S is the classical action of the fluid configuration ϕ, which measures the displacement of fluid elements x_i from their undisturbed positions. Three example configurations are shown in Fig. 3.

We need to know how the phase $S[\phi]$ changes between different curves in Fig. 3. We can understand this by analogy with a free quantum particle with momentum $\hbar p$, which picks up an extra phase $p \times \delta x$ when traveling an extra distance δx. Similarly, the extra phase picked up by a fluid element between two nearby configurations will be the "fluid element momentum" $\times \delta\phi(x_i)$. Summing over all fluid elements produces the total phase difference, and since the "momentum density" is peaked at the location of the soliton, for low wavelength phonons (like the red curve) the phase is well approximated by $\Delta \cdot \delta\phi$ evaluated at the soliton location, where Δ is a number (the "weight") characteristic of the soliton.

The linear coupling of the soliton to the fluid displacement shows the soliton acts as a

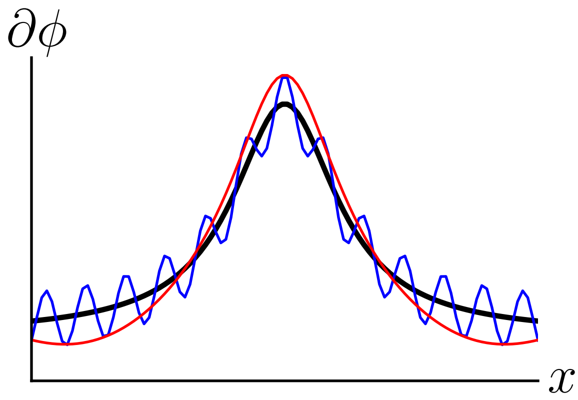


Figure 3: Real space configuration of the soliton (black curve), where $\phi(x_i)$ measures the deviation of a fluid element x_i from its undisturbed position. The vertical axis, $\partial\phi \sim [\phi(x_{i+1}) - \phi(x_i)/\delta x_i]$ is thus the density deviation. The blue (red) curve shows a short (long) wavelength disturbance about the soliton.

potential scatterer for the phonons. Consequently, the soliton propagation amplitude is suppressed by the amplitude for the phonon scattering. Each mode contributes suppresses the action by a finite amount, but when we multiply together the factors for all the scattered modes we find a power law suppression $(v_S t/l_S)^{-\Delta^2/2\pi}$. Here $v_S t$ is the longest wavelength allowed, set by the distance the soliton travels during its existence t, and the short distance cutoff is the width of the soliton. The motion of the soliton contributes a phase $e^{-iE(q)t}$ reflecting its particle–like character, and shifts the power law singularity to the threshold $E(q)$ when we go to Fourier space.

At the phonon threshold $\epsilon(q)$ the physical mechanism is identical (although the theoretical tools for the calculation are slightly different), with a single phonon of energy $\epsilon(q)$ carrying almost all the energy, which scatters low energy phonons and produces the associated power law singularity in the propagation amplitude. The singularities at the phonon and soliton thresholds are the main results of my paper [1].

The mechanism of a local potential causing a power law suppression of matrix elements in a many body system with the system size, due to a proliferation of low energy density waves, was discovered by Phil Anderson [2], who called it the “orthogonality catastrophe”. We might wonder why this physics does not seem to matter for the lineshape in higher dimensions. The reason is that if the scatterer can appreciably recoil, the singularity is washed out. Hence the singularity only remains if the recoil is constrained, either by an infinitely massive scatterer (e.g. a hole in a deep band), or the tightly constricted phase space of 1D.

3 Outlook: Finite T and Quantum Hall Edges

This year the first experiments have seen a finite *linewidth* in a 1D gas of cold ^{87}Rb atoms [3], although the difficulty of working at a small but finite temperature means that the power law singularities have not been directly measured. This presents us with the challenge to extend our calculations to nonzero temperatures.

A particularly intriguing experimental setting is the edge of Quantum Hall droplets. The incompressibility of the Quantum Hall state means that its low energy excitations are edge waves, as illustrated in Fig. 4. Can quantum mechanical solitons exist, and be observed, here? We now know that the smoking gun of solitons is a power law singularity in the density response, if it can be resolved with sufficient precision.

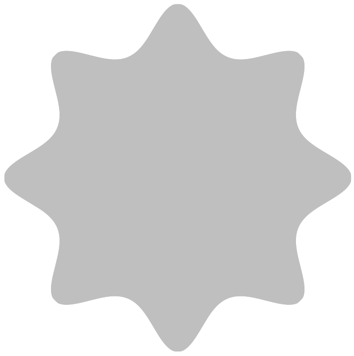


Figure 4: A low energy excitation of an incompressible 2D quantum Hall droplet is a 1D edge wave. The magnetic field, and transverse electric field that confines the droplet, causes the deformation to rotate. Do nonlinear effects (e.g. a nonuniform electric field) result in solitons at the edge?

References

[1] Tom Price and Austen Lamacraft, *Fine Structure of the Phonon in One Dimension From Quantum Hydrodynamics*, Phys. Rev. B, 90:241415, 2014.

[2] P.W. Anderson, *Infrared Catastrophe in Fermi Gases With Local Scattering Potentials*, Phys. Rev. Lett., 18:1049, 1967.

[3] N. Fabbri, M. Panfil, D. Clément, L. Fallani, M. Inguscio, C. Fort, and J.–S. Caux., *Dynamical Structure Factor of One Dimensional Bose Gases: Experimental Signatures of Beyond Luttinger Liquid Physics*, Phys. Rev. A, 91:043617, 2015.